Tulczyjew's Triplet for Lie Groups II: Dynamics

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Abstract

Taking configuration space as a Lie group, the trivialized Euler-Lagrange and Hamilton's equations are obtained and presented as Lagrangian submanifolds of the trivialized Tulczyjew's symplectic space. Euler-Poincaré and Lie-Poisson equations are presented as Lagrangian submanifolds of the reduced Tulczyjew's symplectic space. Tulczyjew's generalized Legendre transformations for trivialized and reduced dynamics are constructed.

Key words Trivialized Euler-Lagrange equations, trivialized Hamilton's equations, Euler-Poincaré equations, Lie-Poisson equations, Morse families, Tulczyjew's triplet, Legendre transformation, Lagrangian submanifold, diffeomorphisms group.
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1 Introduction

Let $\mathcal{Q}$ be the configuration space of a mechanical system. The two wings of the Tulczyjew triplet

$$\begin{tikzcd}
T^*T\mathcal{Q} \arrow[dr, shift left, "\pi_{T\mathcal{Q}}"] & & TT^*\mathcal{Q} \arrow[ll, "\alpha_{\mathcal{Q}}"'] \arrow[dl, "T\pi_{\mathcal{Q}}"] \arrow[dr, "\tau_{T^*\mathcal{Q}}"'] \arrow[rr, "\Omega^\flat_{T^*\mathcal{Q}}"] & & T^*T^*\mathcal{Q} \arrow[dl, shift left, "\pi_{T^*\mathcal{Q}}"] \\
& T\mathcal{Q} \arrow[ul, shift left, "dL"] & & T^*\mathcal{Q} \arrow[ur, shift left, "-dH"] &
\end{tikzcd} \tag{1}$$

defines two different special symplectic structures for Tulczyjew's symplectic space $TT^*\mathcal{Q}$. A Lagrangian L on $T\mathcal{Q}$ (or a Hamiltonian H on $T^*\mathcal{Q}$) generates a Lagrangian submanifold $\mathcal{S}_{TT^*\mathcal{Q}}$ of $TT^*\mathcal{Q}$. Legendre transformation is, then, a transformation between realizations of the same Lagrangian submanifold with two different functions, whose Hessians may be degenerate, and special symplectic structures.

In [14], the right and left global trivializations of Tulczyjew's triplet (1) were adapted for Lie groups

$$\begin{tikzcd}
{}^1T^*TG \arrow[dr, "{}^1\pi_{G \circledS \mathfrak{g}}"'] & & {}^1TT^*G \arrow[ll, "{}^1\bar{\sigma}_G"'] \arrow[rr, "{}^1\Omega^\flat_{G \circledS \mathfrak{g}^*}"] \arrow[dl, "{}^1T\pi_G"] \arrow[dr, "{}^1\tau_{G \circledS \mathfrak{g}^*}"'] & & {}^1T^*T^*G \arrow[dl, "{}^1\pi_{G \circledS \mathfrak{g}}"] \\
& G \circledS \mathfrak{g} & & G \circledS \mathfrak{g}^* &
\end{tikzcd} \tag{2}$$

where $\mathfrak{g}$ is Lie algebra of the group G, $\mathfrak{g}^*$ is the dual of $\mathfrak{g}$, the superscript 1 denotes the global trivialization of the first kind that lifts the Lie group action to iterated bundles.

In this work, we shall study the Lagrangian dynamics on the global trivialization $TG \simeq G \circledS \mathfrak{g}$ and the Hamiltonian dynamics on $T^*G \simeq G \circledS \mathfrak{g}^*$. We shall present trivialized Euler-Lagrange and trivialized Hamilton's equations as Lagrangian submanifolds of the trivialized Tulczyjew's symplectic space ${}^1TT^*G$. We shall then obtain Legendre and inverse Legendre transformations and, arrive at Morse families on $G \circledS (\mathfrak{g} \times \mathfrak{g}^*)$ with fibrations over $\mathfrak{g}^*$ and $\mathfrak{g}$, respectively.

For right invariant Lagrangian and Hamiltonian dynamics, we shall present Euler-Poincaré and Lie-Poisson dynamics as Lagrangian submanifolds of the reduced Tulczyjew's symplectic space $\mathfrak{z}_d = \mathcal{O}_\lambda \times \mathfrak{g}^* \times \mathfrak{g}$ of the reduced trivialized triplet

$$\begin{tikzcd}
\mathcal{O}_\lambda \times \mathfrak{g} \times \mathfrak{g}^* \arrow[dr, "{}^1\pi^{G\backslash}_{G \circledS \mathfrak{g}}"'] & & \mathcal{O}_\lambda \times \mathfrak{g}^* \times \mathfrak{g} \arrow[ll, "{}^1\bar{\sigma}^{G\backslash}_G"'] \arrow[rr, "{}^1\Omega^{G\backslash}_{G \circledS \mathfrak{g}}"] \arrow[dl, "{}^1T\pi^{G\backslash}_G"] \arrow[dr, "{}^1\tau^{G\backslash}_{G \circledS \mathfrak{g}^*}"'] & & \mathcal{O}_\lambda \times \mathfrak{g}^* \times \mathfrak{g} \arrow[dl, "{}^1\pi^{G\backslash}_{G \circledS \mathfrak{g}^*}"] \\
& \mathfrak{g} & & \mathfrak{g}^* &
\end{tikzcd} \tag{3}$$

which will be achieved by means of the Lagrange-Dirac derivative $\mathfrak{d}l : \mathfrak{g} \to \mathfrak{z}_d$ and the Hamilton-Dirac derivative $\mathfrak{d}h : \mathfrak{g} \to \mathfrak{z}_d$ on the reduced Lagrangian and Hamiltonian functions, respectively. The peculiarity of the diagram (3) is that

the right and left wings are not special symplectic structures in the usual classical sense. We shall replace $\mathfrak{z}_l$ with $T^*\mathfrak{g}$ and $\mathfrak{z}_h$ with $T^*\mathfrak{g}^*$ to solve this problem. It turns out that, in the reduced diagram (3), the Morse families are defined on $\mathfrak{g} \times \mathfrak{g}^*$.

In the next section, we shall briefly review the Legendre transformation in the sense of Tulczyjew, and recall the definitions of special symplectic structures and Morse families. In section 3, we shall derive trivialized Euler-Lagrange, trivialized Hamilton's, Euler-Poincaré and Lie-Poisson equations. In section 4, geometry of the trivialized Tulczyjew's triplet in diagram (2) will be summarized. We shall represent trivialized Euler-Lagrange and trivialized Hamilton's equations as Lagrangian submanifolds of ${}^1TT^*G$. Legendre transformations of trivialized dynamics will be established. In section 5, we shall start with the reduced Tulczyjew's triplet in diagram (3) and present Euler-Poincaré and Lie-Poisson dynamics as Lagrangian submanifolds of $\mathfrak{z}_d$. We shall then establish the Legendre transformations of reduced dynamics. In the last section, we shall present the example where G is the group of diffeomorphisms on a manifold.

2 Tulczyjew's Construction of the Legendre Transformation

2.1 Special Symplectic Structures

Let $\mathcal{P}$ be a symplectic manifold carrying an exact symplectic two form $\Omega_{\mathcal{P}} = d\vartheta_{\mathcal{P}}$. A special symplectic structure is a quintuple $(\mathcal{P}, \pi^{\mathcal{P}}_{\mathcal{M}}, \mathcal{M}, \vartheta_{\mathcal{P}}, \chi)$ where $\pi^{\mathcal{P}}_{\mathcal{M}} : \mathcal{P} \to \mathcal{M}$ is a fibre bundle and $\chi : \mathcal{P} \to T^*\mathcal{M}$ is a fiber preserving symplectic diffeomorphism such that $\chi^*\theta_{T^*\mathcal{M}} = \vartheta_{\mathcal{P}}$ for $\theta_{T^*\mathcal{M}}$ being the canonical one-form on $T^*\mathcal{M}$. χ can be characterized uniquely by the condition

$$\left\langle \chi(p), X^{\mathcal{M}}(x) \right\rangle = \left\langle \vartheta_{\mathcal{P}}(p), X^{\mathcal{P}}(p) \right\rangle$$

for each $p \in \mathcal{P}$, $\pi^{\mathcal{P}}_{\mathcal{M}}(p) = x$ and for vector fields $X^{\mathcal{M}}$ and $X^{\mathcal{P}}$ satisfying $\left(\pi^{\mathcal{P}}_{\mathcal{M}}\right)_* X^{\mathcal{P}} = X^{\mathcal{M}}$ [21, 30, 33]. A real valued function F on the base manifold $\mathcal{M}$ defines a Lagrangian submanifold

$$\mathcal{S}_{\mathcal{P}} = \left\{ p \in \mathcal{P} : d\left(F \circ \pi^{\mathcal{P}}_{\mathcal{M}}\right)(p) = \vartheta_{\mathcal{P}}(p) \right\} \tag{4}$$

of the underlying symplectic manifold $(\mathcal{P}, \Omega_{\mathcal{P}} = d\vartheta_{\mathcal{P}})$. The function F together with a special symplectic structure $(\mathcal{P}, \pi^{\mathcal{P}}_{\mathcal{M}}, \mathcal{M}, \vartheta_{\mathcal{P}}, \chi)$ are called a generating family for the Lagrangian submanifold $\mathcal{S}_{\mathcal{P}}$. Since χ is a symplectic diffeomorphism, it maps $\mathcal{S}_{\mathcal{P}}$ to the image space $im\,(dF)$ of the exterior derivative of F, which is a Lagrangian submanifold of $T^*\mathcal{M}$.

2.2 Morse Families

Let $\left(\mathcal{P}, \pi^{\mathcal{P}}_{\mathcal{M}}, \mathcal{M}\right)$ be a fibre bundle. The vertical bundle $V\mathcal{P}$ over $\mathcal{P}$ is the space of vertical vectors $U \in T\mathcal{P}$ satisfying $T\pi^{\mathcal{P}}_{\mathcal{M}}(U) = 0$. The conormal bundle of VP is defined by

$$V^0\mathcal{P} = \left\{ \alpha \in T^*\mathcal{P} : \langle \alpha, U \rangle = 0, \forall U \in V\mathcal{P} \right\}.$$

Let E be a real-valued function on $\mathcal{P}$, then the image $im\,(dE)$ of its exterior derivative is a subspace of $T^*\mathcal{P}$. We say that E is a Morse family (or an energy function) if

$$T_z im\,(dE) + T_z V^0\mathcal{P} = TT^*\mathcal{P}, \tag{5}$$

for all $z \in im\,(dE) \cap V^0\mathcal{P}$, [5, 22, 33, 34, 35, 37]. In local coordinates (x^a, r^i) on the total space $\mathcal{P}$ induced from the coordinates (x^a) on $\mathcal{M}$, the requirement in Eq.(5) reduces to the condition that the rank of the matrix

$$\begin{pmatrix} \dfrac{\partial^2 E}{\partial x^a \partial x^b} & \dfrac{\partial^2 E}{\partial x^a \partial r^i} \end{pmatrix}$$

be maximal. A Morse family E on the smooth bundle $\left(\mathcal{P}, \pi^{\mathcal{P}}_{\mathcal{M}}, \mathcal{M}\right)$ generates an immersed Lagrangian submanifold

$$\mathcal{S}_{T^*\mathcal{M}} = \left\{\lambda_M \in T^*\mathcal{M} : T^*\pi^{\mathcal{P}}_{\mathcal{M}}(\lambda_{\mathcal{M}}) = dE\,(p)\right\} \tag{6}$$

of $(T^*\mathcal{M}, \Omega_{T^*\mathcal{M}})$. Note that, in the definition of $\mathcal{S}_{T^*\mathcal{M}}$, there is an intrinsic requirement that $\pi^{\mathcal{P}}_{\mathcal{M}}\,(p) = \pi_{T^*\mathcal{M}}\,(\lambda_{\mathcal{M}})$.

2.3 The Legendre Transformation

Let $(\mathcal{P}, \Omega_{\mathcal{P}} = d\vartheta_{\mathcal{P}})$ be an exact symplectic manifold, and $(\mathcal{P}, \pi^{\mathcal{P}}_{\mathcal{M}}, \mathcal{M}, \vartheta_{\mathcal{P}}, \chi)$ be a special symplectic structure. A function F on $\mathcal{M}$ defines a Lagrangian submanifold $\mathcal{S}_{\mathcal{P}} \subset \mathcal{P}$ as described in Eq.(4). If $\mathcal{S}_{\mathcal{P}} = im\,(\Upsilon)$ is the image of a section Υ of $\left(\mathcal{P}, \pi^{\mathcal{P}}_{\mathcal{M}}, \mathcal{M}\right)$ then we have $\chi \circ \Upsilon = dF$. Assume that $(\mathcal{P}, \pi^{\mathcal{P}}_{\mathcal{M}'}, \mathcal{M}', \vartheta'_{\mathcal{P}}, \chi')$ is another special symplectic structure associated to the underlying symplectic space $(\mathcal{P}, \Omega_{\mathcal{P}})$. Then, from the diagram

$$\begin{tikzcd}
T^*\mathcal{M} \arrow[dr, shift left, "\pi_{\mathcal{M}}"] & & \mathcal{P} \arrow[ll, "\chi"'] \arrow[rr, "\chi'"] \arrow[dl, shift left, "\pi^{\mathcal{P}}_{\mathcal{M}}"] \arrow[dr, shift left, "\pi^{\mathcal{P}}_{\mathcal{M}'}"] & & T^*\mathcal{M}' \arrow[dl, shift left, "\pi_{\mathcal{M}'}"] \\
& \mathcal{M} \arrow[ul, shift left, "dF"] \arrow[ur, shift left, "\Upsilon"] & & \mathcal{M}' \arrow[ul, shift left, "\Upsilon'"] \arrow[ur, shift left, "dF'"] &
\end{tikzcd} \tag{7}$$

it follows that the difference $\vartheta_{\mathcal{P}} - \vartheta'_{\mathcal{P}}$ of one-forms must be closed in order to satisfy $\Omega_{\mathcal{P}} = d\vartheta_{\mathcal{P}} = d\vartheta'_{\mathcal{P}}$. When the difference is exact, there exists a function Δ on P satisfying $d\Delta = \vartheta_{\mathcal{P}} - \vartheta'_{\mathcal{P}}$. If $\mathcal{S}_{\mathcal{P}}$ is the image of a section Υ' of the fibration $\left(\mathcal{P}, \pi^{\mathcal{P}}_{\mathcal{M}'}, \mathcal{M}'\right)$, then the function

$$F' = \left(F \circ \pi^{\mathcal{P}}_{\mathcal{M}} + \Delta\right) \circ \Upsilon' \tag{8}$$

generates the Lagrangian submanifold $\mathcal{S}_{\mathcal{P}}$ [32, 34, 35]. This is the Legendre transformation. If, finding a global section Υ' of $\pi^{\mathcal{P}}_{\mathcal{M}'}$ satisfying $im(\Upsilon') = \mathcal{S}_{\mathcal{P}}$ is not

possible, the Legendre transformation is not immediate. In this case, define the Morse family

$$E = F \circ \pi^{\mathcal{P}}_{\mathcal{M}} + \Delta \tag{9}$$

on a smooth subbundle of $\left(\mathcal{P}, \pi^{\mathcal{P}}_{\mathcal{M}'}, \mathcal{M}'\right)$, where E satisfies the requirement (5) of being a Morse family. Then, E generates a Lagrangian submanifold $\mathcal{S}_{T^*\mathcal{M}'}$ on $T^*\mathcal{M}'$ as described in Eq.(6). The inverse of χ' maps $\mathcal{S}_{T^*\mathcal{M}'}$ to $\mathcal{S}_{\mathcal{P}}$ bijectively, that is $\mathcal{S}_{\mathcal{P}} = (\chi')^{-1}(\mathcal{S}_{T^*\mathcal{M}'})$.

2.4 The Classical Tulczyjew's Triplet

In this section, we will choose the symplectic manifold $(\mathcal{P}, \Omega_{\mathcal{P}} = d\vartheta_{\mathcal{P}})$, in the diagram (7), to be the Tulczyjew's symplectic space $(TT^*\mathcal{Q}, \Omega_{TT^*\mathcal{Q}})$. Here, $\Omega_{TT^*\mathcal{Q}}$ is the symplectic two-form with two potential one-forms ϑ_1 and ϑ_2 obtained by derivations of canonical one-form $\theta_{T^*\mathcal{Q}}$ and the symplectic-two-form $\Omega_{T^*\mathcal{Q}}$ on $T^*\mathcal{Q}$, respectively. The resulting special symplectic structures

$$\begin{tikzcd} T^*T\mathcal{Q} \arrow[dr, shift left, "\pi_{T\mathcal{Q}}"] & & TT^*\mathcal{Q} \arrow[ll, "\alpha_{\mathcal{Q}}"'] \arrow[rr, "\Omega^{\flat}_{T^*\mathcal{Q}}"] \arrow[dl, "T\pi_{\mathcal{Q}}"] \arrow[dr, "\tau_{T^*\mathcal{Q}}"] & & T^*T^*\mathcal{Q} \arrow[dl, shift left, "\pi_{T^*\mathcal{Q}}"] \\ & T\mathcal{Q} \arrow[ul, shift left, "dL"] & & T^*\mathcal{Q} \arrow[ur, shift left, "-dH"] & \end{tikzcd} \tag{10}$$

where, the musical isomorphism $\Omega^{\flat}_{T^*\mathcal{Q}}$ is induced from $\Omega_{T^*\mathcal{Q}}$, and $\alpha_{\mathcal{Q}}$ is a diffeomorphism constructed as a *dual* of canonical involution of $TT\mathcal{Q}$. They satisfy

$$\left(\Omega^{\flat}_{T^*\mathcal{Q}}\right)^* \theta_{T^*T^*\mathcal{Q}} = \vartheta_1, \quad \alpha^*_{\mathcal{Q}}\theta_{T^*T\mathcal{Q}} = \vartheta_2, \tag{11}$$

where $\theta_{T^*T^*\mathcal{Q}}$ and $\theta_{T^*T\mathcal{Q}}$ canonical one-forms on the cotangent bundles $T^*T^*\mathcal{Q}$ and $T^*T\mathcal{Q}$, respectively.

The generalized Legendre transformation of Lagrangian dynamics on the tangent bundle $T\mathcal{Q}$ can now be constructed as follows: First, present the dynamics as the Lagrangian submanifold of the Tulczyjew's symplectic space $TT^*\mathcal{Q}$. Take the image $im\,(dL)$ of exterior derivative dL which is a Lagrangian submanifold of $T^*T\mathcal{Q}$. Map $im\,(dL)$ by the symplectic diffeomorphism $\alpha_{\mathcal{Q}}$ to a Lagrangian

submanifold $\mathcal{S}_{TT^*\mathcal{Q}}$ of $TT^*\mathcal{Q}$. Alternatively, use the equality

$$(T\pi_{\mathcal{Q}})^* \, dL = \vartheta_2 \tag{12}$$

to obtain $\mathcal{S}_{TT^*\mathcal{Q}}$. Next, generate the same Lagrangian submanifold $\mathcal{S}_{TT^*\mathcal{Q}}$ from the right wing (the Hamiltonian side) of the triplet (10). To achieve this, use a Morse family $E^{L\to H}$ defined on the Pontryagin bundle

$$P\mathcal{Q} = T\mathcal{Q} \times_{\mathcal{Q}} T^*\mathcal{Q}$$

over $T^*\mathcal{Q}$. The Morse family $E^{L\to H}$ defines a Lagrangian submanifold $\mathcal{S}_{T^*T^*\mathcal{Q}}$ of $T^*T^*\mathcal{Q}$. The symplectic diffeomorphism $\Omega^{\flat}_{T^*\mathcal{Q}}$ maps $\mathcal{S}_{T^*T^*\mathcal{Q}}$ to the Lagrangian submanifold $\mathcal{S}_{TT^*\mathcal{Q}}$ obtained by means of the Lagrangian function L. This completes the construction of the Legendre transformation.

For a non-degenerate Lagrangian, the Morse family $E^{L\to H}$ on $P\mathcal{Q}$ can be reduced to a Hamiltonian function H on $T^*\mathcal{Q}$. For degenerate cases, a reduction of the total space $P\mathcal{Q}$ to a subbundle larger than $T^*\mathcal{Q}$ is possible depending on degeneracy level of Lagrangian function [5].

The inverse Legendre transformation, that is to find a Lagrangian formulation of a Hamiltonian system, can be done pursuing the same understanding. The musical isomorphism $\Omega^{\flat}_{T^*\mathcal{Q}}$ maps the image $-im\,(dH)$ of exterior derivative of a Hamiltonian H on $T^*\mathcal{Q}$ to a Lagrangian submanifold $\mathcal{S}'_{TT^*\mathcal{Q}}$ of the Tulczyjew's symplectic space $TT^*\mathcal{Q}$. $\mathcal{S}'_{TT^*\mathcal{Q}}$ can either be defined by the equality

$$-\left(\tau_{T^*\mathcal{Q}}\right)^* dH = \vartheta_1, \tag{13}$$

or as the image of Hamiltonian vector field $-X_H$. The inverse Legendre transformation of the dynamics is meant to generate $\mathcal{S}'_{TT^*\mathcal{Q}}$ by a generating family over the tangent bundle $T\mathcal{Q}$. This can be done with a Morse family $E^{H\to L}$ on the Pontryagin bundle $P\mathcal{Q}$ with fibration over $T\mathcal{Q}$. The Morse family $E^{H\to L}$ defines a Lagrangian submanifold $\mathcal{S}'_{T^*T\mathcal{Q}}$ of $T^*T\mathcal{Q}$ and, the symplectic diffeomorphism $\alpha_{\mathcal{Q}}$ maps $\mathcal{S}'_{T^*T\mathcal{Q}}$ to $\mathcal{S}'_{TT^*\mathcal{Q}}$.

In finite dimensions, introducing the coordinates $(\mathbf{q}, \mathbf{p}; \dot{\mathbf{q}}, \dot{\mathbf{p}})$ on $TT^*\mathcal{Q}$ induced

from Darboux' coordinates $(\mathbf{q},\mathbf{p})$ on $T^*\mathcal{Q}$, one finds the symplectomorphisms

$$\alpha_{\mathcal{Q}}(\mathbf{q},\mathbf{p};\dot{\mathbf{q}},\dot{\mathbf{p}}) = (\mathbf{q},\dot{\mathbf{q}};\dot{\mathbf{p}},\mathbf{p}), \quad \Omega^{\flat}_{T^*\mathcal{Q}}(\mathbf{q},\mathbf{p};\dot{\mathbf{q}},\dot{\mathbf{p}}) = (\mathbf{q},\mathbf{p},-\dot{\mathbf{p}},\dot{\mathbf{q}}),$$

and the potential one-forms

$$\vartheta_1 = \dot{\mathbf{p}}\cdot d\mathbf{q} - \dot{\mathbf{q}}\cdot d\mathbf{p}, \quad \vartheta_2 = \dot{\mathbf{p}}\cdot d\mathbf{q} + \mathbf{p}\cdot d\dot{\mathbf{q}}, \tag{14}$$

where the difference $\vartheta_2 - \vartheta_1$ is the exact one-form $d(\mathbf{p}\cdot\dot{\mathbf{q}})$ on $TT^*\mathcal{Q}$. The Lagrangian submanifold $S_{TT^*\mathcal{Q}}$, defined in Eq.(12), is

$$\nabla_q L(\mathbf{q},\dot{\mathbf{q}}) = \dot{\mathbf{p}}, \quad \nabla_{\dot{q}} L(\mathbf{q},\dot{\mathbf{q}}) = \mathbf{p},$$

which can be written as a second order Euler-Lagrange equation $d(\nabla_{\dot{q}}L)/dt = \nabla_q L$. The energy function

$$E^{L\to H}(\mathbf{q},\mathbf{p},\dot{\mathbf{q}}) = \mathbf{p}\cdot\dot{\mathbf{q}} + L(\mathbf{q},\dot{\mathbf{q}}),$$

satisfies the requirements, given in Eq.(5), of being a Morse family on the Pontryagin bundle $T\mathcal{Q}\times T^*\mathcal{Q}$ over the cotangent bundle $T^*\mathcal{Q}$. Hence, $E^{L\to H}$ generates a Lagrangian submanifold $\mathcal{S}_{T^*T^*\mathcal{Q}}$ of $T^*T^*\mathcal{Q}$ as defined in Eq.(6). In coordinates $(\mathbf{q},\mathbf{p},\mathbf{P}_q,\mathbf{P}_p)$ of $T^*T^*\mathcal{Q}$, $\mathcal{S}_{T^*T^*\mathcal{Q}}$ is given by

$$\mathbf{P}_q = \nabla_q E^{L\to H} = \nabla_q L, \quad \mathbf{P}_p = \nabla_p E^{L\to H} = \dot{\mathbf{q}}, \quad \mathbf{0} = \nabla_{\dot{q}} E^{L\to H} = \mathbf{p} - \nabla_{\dot{q}} L.$$

The inverse musical isomorphism $\Omega^{\sharp}_{T^*\mathcal{Q}}$ maps $\mathcal{S}_{T^*T^*\mathcal{Q}}$ to $\mathcal{S}_{TT^*\mathcal{Q}}$. When the Lagrangian function is non-degenerate, then the Morse family reduces to the Hamiltonian function

$$H(\mathbf{q},\mathbf{p}) = \mathbf{p}\cdot\dot{\mathbf{q}}(\mathbf{q},\mathbf{p}) + L(\mathbf{q},\dot{\mathbf{q}}(\mathbf{q},\mathbf{p}))$$

on $T^*\mathcal{Q}$. In coordinates $(\mathbf{q},\mathbf{p};\dot{\mathbf{q}},\dot{\mathbf{p}})$ on $TT^*\mathcal{Q}$, the Lagrangian submanifold $S'_{TT^*\mathcal{Q}}$, defined in Eq.(13), is the Hamilton's equations

$$\dot{\mathbf{q}} = \nabla_p H(\mathbf{q},\mathbf{p}), \quad \dot{\mathbf{p}} = -\nabla_q H(\mathbf{q},\mathbf{p}). \tag{15}$$

The Morse family

$$E^{H\to L}(\mathbf{q},\mathbf{p},\dot{\mathbf{q}}) = -\mathbf{p}\cdot\dot{\mathbf{q}} + H(\mathbf{q},\mathbf{p})$$

on the Pontryagin bundle $T\mathcal{Q}\times T^*\mathcal{Q}$ over $T\mathcal{Q}$ defines the Lagrangian submanifold $S'_{T^*T\mathcal{Q}}$ of $T^*T\mathcal{Q}$. In coordinates $\left(\mathbf{q},\dot{\mathbf{q}},\mathbf{P}_q,\mathbf{P}_{\dot{q}}\right)$ of $T^*T\mathcal{Q}$, $\mathcal{S}'_{T^*T\mathcal{Q}}$ is given by

$$\mathbf{P}_q=\nabla_q E^{H\rightarrow L}=\nabla_q H,\quad \mathbf{P}_{\dot{q}}=\nabla_{\dot{q}} E^{H\rightarrow L}=-\mathbf{p},\quad \mathbf{0}=\nabla_p E^{H\rightarrow L}=\nabla_p H-\dot{\mathbf{q}}.$$

The inverse $\alpha_{\mathcal{Q}}^{-1}$ of the isomorphism $\alpha_{\mathcal{Q}}$ maps $S'_{T^*T\mathcal{Q}}$ to $S'_{TT^*\mathcal{Q}}$. When the Hamiltonian function is non-degenerate, then the Morse family reduces to the non-degenerate Lagrangian

$$L\left(\mathbf{q},\dot{\mathbf{q}}\right)=-\mathbf{p}\left(\mathbf{q},\dot{\mathbf{q}}\right)\cdot\dot{\mathbf{q}}+H\left(\mathbf{q},\mathbf{p}\left(\mathbf{q},\dot{\mathbf{q}}\right)\right)$$

on $T\mathcal{Q}$.

3 Dynamics on Lie Groups

3.1 Notations

G is a Lie group. Its Lie algebra $\mathfrak{g} \simeq T_e G$ is assumed to be reflexive. The dual of $\mathfrak{g}$ is $\mathfrak{g}^* = Lie^*(G) \simeq T_e^* G$. Throughout the work, we shall designate

$$g, h \in G, \quad \xi, \eta, \zeta \in \mathfrak{g}, \quad \mu, \nu, \lambda \in \mathfrak{g}^*. \tag{16}$$

For a tensor field which is either right or left invariant, we shall use $V_g \in T_g G$ or $\alpha_g \in T_g^* G$ etc... For an arbitrary manifold $\mathcal{M}$, we shall use

$$u, v \in \mathcal{M}, \quad V_u, U_u \in T_u \mathcal{M}, \quad \alpha_u, \beta_u, \gamma_u \in T_u^* \mathcal{M} \tag{17}$$

to denote vectors and one-forms over specific points. We shall denote left and right multiplications on G by L_g and R_g, respectively. The right inner automorphism

$$I_g = L_{g^{-1}} \circ R_g \tag{18}$$

will be a right action of G on G satisfying $I_g \circ I_h = I_{hg}$. The *right* adjoint action $Ad_g = T_e I_g$ of G on $\mathfrak{g}$ is defined as the tangent map of I_g at the identity $e \in G$. The infinitesimal *right* adjoint representation $ad_\xi \eta$ is $[\xi, \eta]_{\mathfrak{g}}$ and it is defined as the derivative of Ad_g at the identity. A right invariant vector field X_ξ^G on G can be obtained by right translation

$$X_\xi^G(g) = T_e R_g \xi \tag{19}$$

of $\xi \in \mathfrak{g}$ for each $g \in G$. The identity

$$[\xi, \eta] = \left[X_\xi^G, X_\eta^G\right]_{JL} \tag{20}$$

gives the isomorphism between $\mathfrak{g}$ and the space $\mathfrak{X}^R(G)$ of right invariant vector fields endowed with the Jacobi-Lie bracket. The coadjoint action Ad_g^* of G on the dual $\mathfrak{g}^*$ of the Lie algebra $\mathfrak{g}$ is a right representation and is the linear algebraic dual of $Ad_{g^{-1}}$, namely,

$$\left\langle Ad_g^* \mu, \xi \right\rangle = \left\langle \mu, Ad_{g^{-1}} \xi \right\rangle \tag{21}$$

holds for all $\xi \in \mathfrak{g}$ and $\mu \in \mathfrak{g}^*$. The infinitesimal coadjoint action ad^*_ξ of $\mathfrak{g}$ on $\mathfrak{g}^*$ is the linear algebraic dual of ad_ξ. Note that, the infinitesimal generator of the coadjoint action Ad^*_g is minus the infinitesimal coadjoint action ad^*_ξ, that is, if $g^t \subset G$ is a curve passing through the identity in the direction of $\xi \in \mathfrak{g}$, then

$$\left.\frac{d}{dt}\right|_{t=0} Ad^*_{g^t}\mu = -ad^*_\xi \mu. \tag{22}$$

The *right* trivialization maps on TG and T^*G are defined to be

$$tr^R_{TG} \;:\; TG \to G \circledS \mathfrak{g} : U_g \to \left(g, T_g R_{g^{-1}} U_g\right), \tag{23}$$

$$tr^R_{T^*G} \;:\; T^*G \to G \circledS \mathfrak{g}^* : \alpha_g \to \left(g, T^*_e R_g \alpha_g\right). \tag{24}$$

We refer to [14] for further details about the right actions and representations.

3.2 Lagrangian dynamics

For a Lagrangian density $L : TG \to \mathbb{R}$, define the unique function $\bar{L}$ on $G \circledS \mathfrak{g}$ by

$$\bar{L}(g, \xi) = \bar{L} \circ tr^R_{TG}(V_g) = L(V_g), \tag{25}$$

where $\xi = T_g R_{g^{-1}} V_g$. The variation of the fiber (Lie algebra) variable ξ can be done by the reduced variational principle [10, 7, 14, 17, 18, 23, 26]

$$\delta\xi = \dot{\eta} + [\xi, \eta]. \tag{26}$$

Proposition *A Lagrangian density $\bar{L}$ on $G \circledS \mathfrak{g}$ defines the trivialized Euler-Lagrange dynamics*

$$\frac{d}{dt}\frac{\delta \bar{L}}{\delta \xi} = T^*_e R_g \frac{\delta \bar{L}}{\delta g} + ad^*_\xi \frac{\delta \bar{L}}{\delta \xi}. \tag{27}$$

Proof *Using reduced variational principle, one computes*

$$
\begin{aligned}
\delta \int_b^a \bar{L}(g,\xi)\,dt &= \int_b^a \left(\left\langle \frac{\delta \bar{L}}{\delta g}, \delta g \right\rangle_g + \left\langle \frac{\delta \bar{L}}{\delta \xi}, \delta \xi \right\rangle_e \right) dt \\
&= \int_b^a \left(\left\langle \frac{\delta \bar{L}}{\delta g}, \delta g \right\rangle_g + \left\langle \frac{\delta \bar{L}}{\delta \xi}, \dot{\eta} + [\xi, \eta] \right\rangle_e \right) dt \\
&= -\left\langle \frac{\delta \bar{L}}{\delta \xi}, \eta \right\rangle_e \Big|_b^a + \int_b^a \left(\left\langle \frac{\delta \bar{L}}{\delta g}, \delta g \right\rangle_g + \left\langle -\frac{d}{dt}\frac{\delta \bar{L}}{\delta \xi} + ad_\xi^* \frac{\delta \bar{L}}{\delta \xi}, \eta \right\rangle_e \right) dt \\
&= -\left\langle \frac{\delta \bar{L}}{\delta \xi}, T_g R_{g^{-1}} \delta g \right\rangle_e \Big|_b^a + \int_b^a \left\langle \frac{\delta \bar{L}}{\delta g}, \delta g \right\rangle_g + \left\langle ad_\xi^* \frac{\delta \bar{L}}{\delta \xi} - \frac{d}{dt}\frac{\delta \bar{L}}{\delta \xi}, T_g R_{g^{-1}} \delta g \right\rangle_e dt \\
&= -\left\langle T_g^* R_{g^{-1}} \frac{\delta \bar{L}}{\delta \xi}, \delta g \right\rangle_g \Big|_a^b + \int_b^a \left\langle \frac{\delta \bar{L}}{\delta g} + T_g^* R_{g^{-1}} \left(ad_\xi^* \frac{\delta \bar{L}}{\delta \xi} - \frac{d}{dt}\frac{\delta \bar{L}}{\delta \xi} \right), \delta g \right\rangle_g dt.
\end{aligned}
$$

and the conclusion follows if δg vanishes at boundaries.

The trivialized Euler-Lagrange equation (27) is defined over the identity $e \in G$. Because $\delta \bar{L}/\delta g \in T_g^* G$ and the functor $T_e^* R_g$ takes this to the dual space $\mathfrak{g}^* = T_e^* G$. Eq.(27) appeared in [11] with the missing operator $T_e^* R_g$. It also appeared in some recent works [8, 9] on the higher order Lagrangian and Hamiltonian dynamics on trivialized iterated bundles of Lie groups. See also [6].

When the Lagrangian $\bar{L}$ is independent of g, that is, $\bar{L}(g,\xi) = l(\xi)$ and L on TG is right invariant, then Eq.(27) reduces to the Euler-Poincaré equation

$$
ad_\xi^* \frac{\delta l}{\delta \xi} - \frac{d}{dt}\frac{\delta l}{\delta \xi} = 0. \tag{28}
$$

3.3 Hamiltonian dynamics

By pushing forward the canonical one-form θ_{T^*G} and symplectic two-form Ω_{T^*G} on T^*G with the trivialization map $tr_{T^*G}^R$, $G \circledS \mathfrak{g}^*$ can be endowed with an exact symplectic two-form $\Omega_{G \circledS \mathfrak{g}^*} = d\theta_{G \circledS \mathfrak{g}^*}$. If

$$
X_{(\xi,\nu)}^{G \circledS \mathfrak{g}^*}(g,\mu) = \left(T_e R_g \xi, \nu + ad_\xi^{R*} \mu \right),
$$

is a right invariant vector field at the point $(g,\mu) \in G \circledS \mathfrak{g}^*$ generated by the Lie algebra element $(\xi,\nu) \in Lie(G \circledS \mathfrak{g}^*) \simeq \mathfrak{g} \circledS \mathfrak{g}^*$ then, the values of canonical

one-forms $\theta_{G\circledS\mathfrak{g}^*}$ and $\Omega_{G\circledS\mathfrak{g}^*}$ on $X_{(\xi,\nu)}^{G\circledS\mathfrak{g}^*}(g,\mu)$ are [1, 14]

$$\left\langle \theta_{G\circledS\mathfrak{g}^*}, X_{(\xi,\nu)}^{G\circledS\mathfrak{g}^*}\right\rangle (g,\mu) = \langle \mu, \xi\rangle \tag{29}$$

$$\left\langle \Omega_{G\circledS\mathfrak{g}^*}; \left(X_{(\xi,\nu)}^{G\circledS_R\mathfrak{g}^*}, X_{(\eta,\lambda)}^{G\circledS\mathfrak{g}^*}\right)\right\rangle (g,\mu) = \langle \nu, \eta\rangle - \langle \lambda, \xi\rangle + \left\langle \mu, [\xi,\eta]_{\mathfrak{g}}\right\rangle \tag{30}$$

which are considered for linearizations of Hamiltonian systems in [26] and, for higher order dynamics in [8]. Let H be a function on T^*G and define $\bar{H}: G\circledS\mathfrak{g}^* \to \mathbb{R}$ by $\bar{H}\circ tr_{T^*G}^{R} = H$, that is, for $\alpha_g = T_g^* R_{g^{-1}}\mu$, we have $\bar{H}(g,\mu) = H(\alpha_g)$ and Hamilton's equations on $(G\circledS\mathfrak{g}^*, \Omega_{G\circledS\mathfrak{g}^*})$ are

$$i_{X_{\bar{H}}^{G\circledS\mathfrak{g}^*}} \Omega_{G\circledS\mathfrak{g}^*} = -d\bar{H}. \tag{31}$$

Proposition *The Hamiltonian vector field $X_{\bar{H}}^{G\circledS_R\mathfrak{g}^*}$, defined in Eq.(31), is generated by the element*

$$\left(\frac{\delta\bar{H}}{\delta\mu}, -T_e^* R_g\left(\frac{\delta\bar{H}}{\delta g}\right)\right)$$

of the Lie algebra $\mathfrak{g}\circledS\mathfrak{g}^*$ of $G\circledS\mathfrak{g}^*$ and components of $X_{\bar{H}}^{G\circledS_R\mathfrak{g}^*}$ are given by the trivialized Hamilton's equations

$$\frac{dg}{dt} = T_e R_g\left(\frac{\delta\bar{H}}{\delta\mu}\right), \quad \frac{d\mu}{dt} = ad_{\frac{\delta\bar{H}}{\delta\mu}}^* \mu - T_e^* R_g \frac{\delta\bar{H}}{\delta g}. \tag{32}$$

Note that the second term on the right hand side of the second equation in Eq.(32) is a consequence of the semidirect product structure on $G\circledS\mathfrak{g}^*$. Accordingly, if we let $\bar{H}$ to be independent of the group element, that is, $\bar{H}(g,\mu) = h(\mu)$ and H on T^*G is right invariant, then the trivialized Hamilton's equations (32) reduce to

$$\frac{dg}{dt} = T_e R_g\left(\frac{\delta h}{\delta\mu}\right), \quad \frac{d\mu}{dt} = ad_{\frac{\delta h}{\delta\mu}}^* \mu. \tag{33}$$

The canonical Poisson bracket on $G \circledS \mathfrak{g}^*$ is

$$
\begin{aligned}
& \left\{\bar{F}, \bar{K}\right\}_{G \circledS \mathfrak{g}^*}(g, \mu) = \Omega_{G \circledS \mathfrak{g}^*}\left(X_{\bar{F}}^{G \circledS \mathfrak{g}^*}, X_{\bar{K}}^{G \circledS \mathfrak{g}^*}\right)(g, \mu) \\
= \ & \Omega_{G \circledS \mathfrak{g}^*}\left(X^{G \circledS \mathfrak{g}^*}_{\left(\frac{\delta \bar{F}}{\delta \mu}, -T^* R_g \frac{\delta \bar{F}}{\delta g}\right)}, X^{G \circledS \mathfrak{g}^*}_{\left(\frac{\delta \bar{K}}{\delta \mu}, -T^* R_g \frac{\delta \bar{K}}{\delta g}\right)}\right)(g, \mu) \\
= \ & \left\langle T_e^* R_g \frac{\delta \bar{K}}{\delta g}, \frac{\delta \bar{F}}{\delta \mu}\right\rangle - \left\langle T_e^* R_g \frac{\delta \bar{F}}{\delta g}, \frac{\delta \bar{K}}{\delta \mu}\right\rangle + \left\langle \mu, \left[\frac{\delta \bar{F}}{\delta \mu}, \frac{\delta \bar{K}}{\delta \mu}\right]_{\mathfrak{g}}\right\rangle,
\end{aligned}
$$

for two function(al)s $\bar{F}$ and $\bar{K}$ defined on $G \circledS \mathfrak{g}^*$. The Poisson bracket $\{\ ,\ \}_{G \circledS \mathfrak{g}^*}$ is non-degenerate. When $\bar{F}$ and $\bar{K}$ are independent of the group variable $g \in G$, that is, $\bar{F} = f(\mu)$ and $\bar{K} = k(\mu)$, we have the Lie-Poisson bracket

$$
\{f, k\}_{\mathfrak{g}^*}(\mu) = \left\langle \mu, \left[\frac{\delta f}{\delta \mu}, \frac{\delta k}{\delta \mu}\right]_{\mathfrak{g}}\right\rangle \tag{34}
$$

on the dual space $\mathfrak{g}^*$ [3, 18, 23, 19]. This is a manifestation of the fact that the projection $G \circledS \mathfrak{g}^* \to \mathfrak{g}^*$ is the momentum map for the cotangent lifted left action of G on $G \circledS \mathfrak{g}^*$. In this case, the dynamics is driven by the Hamiltonian vector field $X_h^{\mathfrak{g}^*}$ satisfying

$$
\{f, h\}_{\mathfrak{g}^*} = -\left\langle df, X_h^{\mathfrak{g}^*}\right\rangle
$$

for a given Hamiltonian function(al) h on $\mathfrak{g}^*$. More explicitly, the value of Hamiltonian vector field $X_h^{\mathfrak{g}^*}$ at $\mu \in \mathfrak{g}^*$ is defined by the Lie-Poisson equations

$$
\frac{d\mu}{dt} = ad^*_{\frac{\delta h}{\delta \mu}} \mu. \tag{35}
$$

We shall refer to both of the equations in (33) and (35) as Lie-Poisson equations [26].

4 The Trivialized Dynamics

4.1 Trivialization of the Tulczyjew's Triplet

The tangent and cotangent lifts of the group structure on G define group structures on TG and T^*G, respectively. The trivialization maps tr^R_{TG} and $tr^R_{T^*G}$ on TG and T^*G are defined in such a way that they are not only diffeomorphisms but also group isomorphisms [16, 20, 24, 25, 26, 27, 28]. For iterated bundles, with the same understanding of trivializations we obtained [14]

$$
\begin{aligned}
tr^1_{T^*(G\circledS\mathfrak{g})} &: T^*\left(G\circledS\mathfrak{g}\right) \to \left(G\circledS\mathfrak{g}\right)\circledS\left(\mathfrak{g}^*\times\mathfrak{g}^*\right) = {}^1T^*TG &(36)\\
&: \left(\alpha_g,\alpha_\xi\right) \to \left(g,\xi,T_e^*R_g\left(\alpha_g\right)+ad^*_\xi\alpha_\xi,\alpha_\xi\right), \\
tr^1_{T^*(G\circledS\mathfrak{g}^*)} &: T^*T^*G \to \left(G\circledS\mathfrak{g}^*\right)\circledS\left(\mathfrak{g}^*\times\mathfrak{g}\right) = {}^1T^*T^*G &(37)\\
&: \left(\alpha_g,\alpha_\mu\right) \to \Big(g,\mu,T_e^*R_g\left(\alpha_g\right)-ad^*_{\alpha_\mu}\mu,\alpha_\mu\Big), \\
tr^1_{T(G\circledS\mathfrak{g}^*)} &: TT^*G \to \left(G\circledS\mathfrak{g}^*\right)\circledS\left(\mathfrak{g}\circledS\mathfrak{g}^*\right) = {}^1TT^*G &(38)\\
&: \left(V_g,V_\mu\right) \to \Big(g,\mu,TR_{g^{-1}}V_g,V_\mu-ad^*_{TR_{g^{-1}}V_g}\mu\Big).
\end{aligned}
$$

Although, not unique, this way of trivializing iterated bundles enables us to perform reductions of Tulczyjew's triplet.

The symplectic two-forms on the trivialized bundles ${}^1T^*TG$, ${}^1T^*T^*G$ and ${}^1TT^*G$ have been constructed based on the fact that the trivialization maps $tr^1_{T^*(G\circledS\mathfrak{g})}$, $tr^1_{T^*(G\circledS\mathfrak{g}^*)}$ and $tr^1_{T(G\circledS\mathfrak{g}^*)}$ are symplectic diffeomorphisms. The trivialized Tulczyjew's triplet

$$
\begin{tikzcd}
{}^1T^*TG \arrow[dr, "{}^1\pi_{G\circledS\mathfrak{g}}"'] & & {}^1TT^*G \arrow[ll, "{}^1\bar{\sigma}_G"'] \arrow[rr, "{}^1\Omega^\flat_{G\circledS\mathfrak{g}^*}"] \arrow[dl, "{}^1T\pi_G"] \arrow[dr, "{}^1\tau_{G\circledS\mathfrak{g}^*}"'] & & {}^1T^*T^*G \arrow[dl, "{}^1\pi_{G\circledS\mathfrak{g}}"] \\
& G\circledS\mathfrak{g} & & G\circledS\mathfrak{g}^* &
\end{tikzcd}
\tag{39}
$$

consists of trivialized symplectic diffeomorphisms ${}^1\bar{\sigma}_G$ and ${}^1\Omega^\flat_{G\circledS\mathfrak{g}^*}$, and projections

whose local expressions are

$$
\begin{aligned}
{}^1\bar{\sigma}_G &: \quad {}^1TT^*G \to {}^1T^*TG : (g,\mu,\xi,\nu) \to \left(g,\xi,\nu + ad^*_{\xi}\mu,\mu\right), && (40)\\
{}^1\Omega^{\flat}_{G\circledS\mathfrak{g}^*} &: \quad {}^1TT^*G \to {}^1T^*T^*G : (g,\mu,\xi,\nu) \to \left(g,\mu,\nu + ad^*_{\xi}\mu,-\xi\right), && (41)\\
{}^1T\pi_G &: \quad {}^1TT^*G \to G\circledS\mathfrak{g} : (g,\mu,\xi,\nu) \to (g,\xi), && (42)\\
{}^1\pi_{G\circledS\mathfrak{g}^*} &: \quad {}^1T^*T^*G \to G\circledS\mathfrak{g}^* : (g,\mu,\nu,\xi) \to (g,\mu), && (43)\\
{}^1\pi_{G\circledS\mathfrak{g}} &: \quad {}^1T^*TG \to G\circledS\mathfrak{g} : (g,\xi,\mu,\nu) \to (g,\xi). && (44)
\end{aligned}
$$

4.2 Trivialized Tulczyjew's Symplectic Space

Lie algebra of the group ${}^1TT^*G$ is the semi-direct product $(\mathfrak{g}\circledS\mathfrak{g}^*)\circledS(\mathfrak{g}\circledS\mathfrak{g}^*)$. A Lie algebra element

$$(\xi_2,\nu_2,\xi_3,\nu_3) \in (\mathfrak{g}\circledS\mathfrak{g}^*)\circledS(\mathfrak{g}\circledS\mathfrak{g}^*)$$

defines a right invariant vector field on ${}^1TT^*G$ by the tangent lift of right translation in ${}^1TT^*G$. At a point (g,μ,ξ,ν), a right invariant vector field takes the value

$$X^{{}^1TT^*G}_{(\xi_2,\nu_2,\xi_3,\nu_3)} = \left(TR_g\xi_2, \nu_2 + ad^*_{\xi_2}\mu, \xi_3 + [\xi,\xi_2]_{\mathfrak{g}}, \nu_3 + ad^*_{\xi_2}\nu - ad^*_{\xi}\nu_2\right). \tag{45}$$

By requiring the trivialization $tr^1_{TT^*G}$ be a symplectic mapping, we obtain an exact symplectic structure $\Omega_{{}^1TT^*G}$ with two potential one-forms θ_1 and θ_2 on ${}^1TT^*G$. At a point $(g,\mu,\xi,\nu) \in {}^1TT^*G$, the values of the potential one-forms θ_1 and θ_2 on the right invariant vector field of the form of Eq.(45) are

$$
\begin{aligned}
\left\langle \theta_1, X^{{}^1TT^*G}_{(\xi_2,\nu_2,\xi_3,\nu_3)} \right\rangle &= \langle \nu,\xi_2\rangle - \langle \nu_2,\xi\rangle + \left\langle \mu, [\xi,\eta]_{\mathfrak{g}} \right\rangle, && (46)\\
\left\langle \theta_2, X^{{}^1TT^*G}_{(\xi_2,\nu_2,\xi_3,\nu_3)} \right\rangle &= \langle \mu,\xi_3\rangle + \langle \nu,\xi_2\rangle + \left\langle \mu, [\xi,\xi_2]_{\mathfrak{g}} \right\rangle, && (47)
\end{aligned}
$$

respectively. At the same point, the value of symplectic two-form $\Omega_{{}^1TT^*G}$ on two right invariant vector fields is

$$
\begin{aligned}
&\left\langle \Omega_{{}^1TT^*G}; \left(X^{{}^1TT^*G}_{(\xi_2,\nu_2,\xi_3,\nu_3)}, X^{{}^1TT^*G}_{(\bar{\xi}_2,\bar{\nu}_2,\bar{\xi}_3,\bar{\nu}_3)} \right) \right\rangle = \left\langle \nu_3, \bar{\xi}_2 \right\rangle + \left\langle \nu_2, \bar{\xi}_3 \right\rangle - \langle \bar{\nu}_2, \xi_3\rangle - \langle \bar{\nu}_3, \xi_2\rangle \\
&+ \left\langle \nu, \left[\xi_2,\bar{\xi}_2\right]_{\mathfrak{g}} \right\rangle + \left\langle \mu, \left[\xi_3,\bar{\xi}_2\right]_{\mathfrak{g}} + \left[\xi_2,\bar{\xi}_3\right]_{\mathfrak{g}} + \left[\xi, \left[\xi_2,\bar{\xi}_2\right]_{\mathfrak{g}}\right] \right\rangle.
\end{aligned}
$$

Existence of potential one-forms in Eqs.(46) and (47) leads us to define two special symplectic structures

$$\left({}^1TT^*G,\ {}^1\tau_{Gⓢ\mathfrak{g}^*},\ {}^1T^*T^*G,\ \theta_1,\ {}^1\Omega^\flat_{Gⓢ\mathfrak{g}^*} \right) \tag{48}$$
$$\left({}^1TT^*G,\ {}^1T\pi_G,\ {}^1T^*TG,\ \theta_2,\ {}^1\bar{\sigma}_G \right), \tag{49}$$

on the trivialized Tulczyjew's symplectic manifold $({}^1TT^*G, \Omega_{\ {}^1TT^*G})$. The structures in Eqs.(48) and (49) are the right and left wings of the trivialized Tulczyjew's triplet (39), respectively. We refer to [14] for details.

4.3 Trivialized Lagrangian Dynamics as a Lagrangian Submanifold

Proposition *Let $\bar{L}$ be a Lagrangian on $Gⓢ\mathfrak{g}$, then the Lagrangian submanifold $S_{\ {}^1TT^*G}$ defined by the equation*

$$\left({}^1T\pi_G \right)^* d\bar{L} = \theta_2, \tag{50}$$

gives the trivialized Euler-Lagrange equations (27). Here, the projection ${}^1T\pi_G$ is given by Eq.(42) and θ_2 is the one-form in Eq.(47).

Proof *Under the global trivialization ${}^1TT^*G$ of $T\left(Gⓢ\mathfrak{g}^*\right)$, given in Eq.(38), the Lagrangian submanifold described by Eq.(50) becomes*

$$S_{\ {}^1TT^*G} = \left\{ \left(g, \frac{\delta \bar{L}}{\delta \xi}, \xi, T^*R_g \frac{\delta \bar{L}}{\delta g} \right) \in\ {}^1TT^*G : (g,\xi) \in Gⓢ\mathfrak{g} \right\}. \tag{51}$$

To relate this to the trivialized Euler-Lagrange equations (27), we recall the reconstruction mapping

$$\left(tr^1_{T(Gⓢ\mathfrak{g}^*)} \right)^{-1}:\ {}^1TT^*G \to T\left(Gⓢ\mathfrak{g}^*\right) : (g,\mu,\xi,\nu) \to \left(g,\mu, TR_g\xi, \nu + ad^*_\xi \mu \right), \tag{52}$$

computed from Eq.(38). $\left(tr^1_{T(Gⓢ\mathfrak{g}^*)} \right)^{-1}$ maps $S_{\ {}^1TT^*G}$ to the Lagrangian submanifold

$$S_{T(Gⓢ\mathfrak{g}^*)} = \left\{ \left(g, \frac{\delta \bar{L}}{\delta \xi}; TR_g\xi, T^*R_g \frac{\delta \bar{L}}{\delta g} + ad^*_\xi \frac{\delta \bar{L}}{\delta \xi} \right) \in S_{\ {}^1TT^*G} : (g,\xi) \in Gⓢ\mathfrak{g} \right\} \tag{53}$$

of $T\left(G\circledS\mathfrak{g}^{*}\right)$ and this determines the trivialized Euler-Lagrange equations (27).

As mentioned in the context of general theory, an alternative way to obtain $S_{\,^{1}TT^{*}G}$ is to consider a function $\bar{L}$ on $G\circledS\mathfrak{g}$ together with the special symplectic structure (49). This time using the trivialization $tr^{1}_{T^{*}(G\circledS\mathfrak{g})}$ in Eq.(36), we obtain the trivialization of exterior derivative ${}^{1}d\bar{L} := tr^{R}_{T^{*}(G\circledS\mathfrak{g})}\circ d\bar{L}$ which defines, through $im\left({}^{1}d\bar{L}\right)$, the Lagrangian submanifold

$$S_{\,^{1}T^{*}TG} = \left\{\left(g,\xi,T_{e}^{*}R_{g}\frac{\delta\bar{L}}{\delta g}+ad_{\xi}^{*}\frac{\delta\bar{L}}{\delta\xi},\frac{\delta\bar{L}}{\delta\xi}\right)\in\ {}^{1}T^{*}TG:(g,\xi)\in G\circledS\mathfrak{g}\right\} \tag{54}$$

of $\left(\ {}^{1}T^{*}TG,\ {}^{1}\Omega_{T^{*}(G\circledS\mathfrak{g})}\right)$. The inverse ${}^{1}\bar{\sigma}_{G}^{-1}$ of the diffeomorphism ${}^{1}\bar{\sigma}_{G}$ in Eq.(40) takes the Lagrangian submanifold $im\left(\ {}^{1}d\bar{L}\right)$ in Eq.(54) to the Lagrangian submanifold $S_{\,^{1}TT^{*}G}$ in Eq.(51).

4.4 Trivialized Hamiltonian Dynamics as a Lagrangian Submanifold

Proposition *The Lagrangian submanifold defined by the equation*

$$-\left(\ {}^{1}\tau_{G\circledS\mathfrak{g}^{*}}\right)^{*}d\bar{H}=\theta_{1} \tag{55}$$

determines the trivialized Hamilton's equations (32). Here, ${}^{1}\tau_{G\circledS\mathfrak{g}^{*}}$ is the tangent bundle projection and θ_{1} is the one-form in Eq.(46).

Proof *Under the global trivialization ${}^{1}TT^{*}G$ of $TT^{*}G$ given in Eq.(38), the Lagrangian submanifold (55) can be described as*

$$S'_{\,^{1}TT^{*}G} = \left\{\left(g,\mu,\frac{\delta\bar{H}}{\delta\mu},-T^{*}R_{g}\frac{\delta\bar{H}}{\delta g}\right)\in\ {}^{1}TT^{*}G:(g,\mu)\in G\circledS\mathfrak{g}^{*}\right\}. \tag{56}$$

The reconstruction mapping $\left(tr^{1}_{T(G\circledS\mathfrak{g}^{*})}\right)^{-1}$ in Eq.(52) maps $S'_{\,^{1}TT^{*}G}$ to the Lagrangian submanifold

$$S'_{T(G\circledS\mathfrak{g}^{*})} = \left\{\left(TR_{g}\left(\frac{\delta\bar{H}}{\delta\mu}\right),ad^{*}_{\frac{\delta\bar{H}}{\delta\mu}}\mu-TR_{g}^{*}\frac{\delta\bar{H}}{\delta g}\right)\in T\left(G\circledS\mathfrak{g}^{*}\right):(g,\mu)\in G\circledS\mathfrak{g}^{*}\right\} \tag{57}$$

which is the image of Hamiltonian vector field $X_{\bar{H}}^{G\circledS\mathfrak{g}^{*}}$ defined in Eq.(32).

Alternatively, using the trivialization of the exterior derivative

$$-\ {}^{1}d\bar{H} = -tr^{R}_{T^{*}(G \circledS \mathfrak{g}^{*})} \circ d\left(\bar{H}\right)$$

we obtain the Lagrangian submanifold

$$S'_{{}^{1}T^{*}T^{*}G} = \left\{ \left(g, \mu, ad^{*}_{\frac{\delta \bar{H}}{\delta \mu}} \mu - T^{*}_{e} R_{g} \frac{\delta \bar{H}}{\delta g}, -\frac{\delta \bar{H}}{\delta \mu} \right) \in\ {}^{1}T^{*}T^{*}G : (g, \mu) \in G \circledS \mathfrak{g}^{*} \right\} \tag{58}$$

of ${}^{1}T^{*}T^{*}G$. The inverse ${}^{1}\Omega^{\sharp}_{G\circledS\mathfrak{g}^{*}}$ of the isomorphism ${}^{1}\Omega^{\flat}_{G\circledS\mathfrak{g}^{*}}$ maps $S'_{{}^{1}T^{*}T^{*}G}$ to the Lagrangian submanifold $S'_{{}^{1}TT^{*}G}$. This description of $S'_{{}^{1}TT^{*}G}$ is the usual form of Hamilton's equation with respect to the symplectic two-form ${}^{1}\Omega_{G\circledS\mathfrak{g}^{*}}$.

4.5 Legendre Transformation for Trivialized Dynamics

In the previous section, the trivialized Euler-Lagrange equations (27) have been reformulated as the Lagrangian submanifold $S_{{}^{1}TT^{*}G}$ described in Eq.(51). We are now ready to perform the Legendre transformation, that is to describe $S_{{}^{1}TT^{*}G}$ from Hamiltonian side (bundles over $G\circledS\mathfrak{g}^{*}$) of the trivialized Tulczyjew's triplet (39).

Proposition *The Lagrangian dynamics determined by the Lagrangian submanifold $S_{{}^{1}TT^{*}G}$ in Eq.(51) is generated by the Morse family*

$$E^{\bar{L}\to\bar{H}} = \left(\bar{L} \circ\ {}^{1}T\pi_{G}\right) + \Delta = \bar{L}\left(g, \xi\right) - \langle \mu, \xi \rangle \tag{59}$$

defined on the (right) trivialized Pontryagin bundle ${}^{1}PG = G\circledS\left(\mathfrak{g}\times\mathfrak{g}^{*}\right)$ over $G\circledS\mathfrak{g}^{*}$. Here, the function $\Delta = \langle \mu, \xi \rangle$ is defined as to satisfy

$$d\Delta = \theta_{1} - \theta_{2} = -\left\langle \mu, \xi_{3} \right\rangle - \left\langle \nu_{2}, \xi \right\rangle.$$

Remark *The right trivialization of the Pontryagin bundle $PG = TG \times_{G} T^{*}G$ is*

$$\begin{aligned} tr^{1}_{PG} &: \ TG \times_{G} T^{*}G \to G\circledS\left(\mathfrak{g}\times\mathfrak{g}^{*}\right) =:\ {}^{1}PG \\ &: \ \left(V_{g}, \alpha_{g}\right) \to \left(g, T_{g}R_{g^{-1}}V_{g}, T^{*}_{e}R_{g}\alpha_{g}\right). \end{aligned}$$

In [12], the details of the trivialized Pontryagin bundle ${}^{1}PG$ will be presented along

with the implicit trivialized Euler-Lagrange and implicit trivialized Hamiltonian dynamics on ^1PG.

Remark *The potential function* Δ *is the value of canonical one-form* $\theta_{G\circledS\mathfrak{g}^*}$ *on the right invariant vector field* $X_{(\xi,\nu)}^{G\circledS\mathfrak{g}^*}$ *as given in Eq.(29).*

Proof *The Morse family* $E^{\bar{L}\to\bar{H}}$*, in Eq.(59), determines a Lagrangian submanifold* $S_{T^*(G\circledS\mathfrak{g}^*)}$ *which can be described by the equations*

$$\alpha_g = \frac{\delta E^{\bar{L}\to\bar{H}}}{\delta g} = \frac{\delta \bar{L}}{\delta g}, \quad \alpha_\mu = \frac{\delta E^{\bar{L}\to\bar{H}}}{\delta \mu} = -\xi, \quad 0 = \frac{\delta E^{\bar{L}\to\bar{H}}}{\delta \xi} = \frac{\delta \bar{L}}{\delta \xi} - \mu$$

defined on the coordinates (α_g, α_μ) *of* $T^*_{(g,\mu)}(G\circledS\mathfrak{g}^*)$. *The trivialization* $tr^1_{T^*(G\circledS\mathfrak{g}^*)}$ *maps* $S_{T^*(G\circledS\mathfrak{g}^*)}$ *to the Lagrangian submanifold*

$$S_{\,^1T^*T^*G} = \left(g, \frac{\delta \bar{L}}{\delta \xi}, T^*R_g \frac{\delta \bar{L}}{\delta g} - ad^*_\xi \frac{\delta \bar{L}}{\delta \xi}, -\xi\right)$$

of $^1T^*T^*G$. *The musical isomorphism* $^1\Omega^\sharp_{G\circledS\mathfrak{g}^*}$*, in turn, maps* $S_{\,^1T^*T^*G}$ *to the Lagrangian submanifold* $S_{\,^1TT^*G}$ *in Eq.(51).*

Remark *When we have* $\bar{L} = l(\xi)$*, the trivialized Euler-Lagrange equations reduce to Euler-Poincaré equations. In this case, the Legendre transformation is generated by the Morse family*

$$E^{\bar{L}\to\bar{H}} = l(\xi) - \langle \mu, \xi \rangle. \tag{60}$$

The inverse Legendre transformation defines a Lagrangian formulation for the trivialized Hamilton's Eq.(32) which is represented by the Lagrangian submanifold $S'_{\,^1TT^*G}$ described in Eq.(56). The following proposition shows how to find an alternative generating family for $S'_{\,^1TT^*G}$ that will lead to its representation on the Lagrangian side of the triplet (39).

Proposition *The Morse family*

$$E^{\bar{H}\to\bar{L}} = \left(-\bar{H} \circ^1 T\pi_G\right) - \Delta = \langle \mu, \xi \rangle - \bar{H}(g,\mu) \tag{61}$$

defined on the trivialized Pontryagin bundle $^1PG = G\circledS(\mathfrak{g}\times\mathfrak{g}^*)$ *over* $G\circledS\mathfrak{g}$ *determines the Lagrangian submanifold* $S'_{\,^1TT^*G}$ *in Eq.(56).*

Proof *The Lagrangian submanifold* $S_{T^*(G \circledS \mathfrak{g})}$ *of* $T^*(G \circledS \mathfrak{g})$ *defined by the Morse family (61) is given by*

$$\alpha_g = \frac{\delta E^{\bar{H} \to \bar{L}}}{\delta g} = -\frac{\delta \bar{H}}{\delta g}, \quad \alpha_\xi = \frac{\delta E^{\bar{H} \to \bar{L}}}{\delta \xi} = \mu, \quad 0 = \frac{\delta E^{\bar{H} \to \bar{L}}}{\delta \mu} = -\frac{\delta \bar{H}}{\delta \mu} + \xi,$$

where (α_g, α_ξ) *are coordinates on* $T^*_{(g,\xi)}(G \circledS \mathfrak{g})$*. The trivialization* $tr^1_{T^*(G \circledS \mathfrak{g})}$ *in Eq.(36) maps* $S_{T^*(G \circledS \mathfrak{g})}$ *to the Lagrangian submanifold*

$$S_{\ ^1T^*TG} = \left(g, \xi, -T^* R_g \frac{\delta \bar{H}}{\delta g} + ad^*_{\frac{\delta \bar{H}}{\delta \mu}} \mu, \mu \right)$$

of $^1T^*TG$*. The inverse of the isomorphism* $^1\bar{\sigma}_G$ *in Eq.(40) takes* $S_{\ ^1T^*TG}$ *to the Lagrangian submanifold* $S'_{^1TT^*G}$ *in Eq.(56).*

Remark *When* $\bar{H} = h(\mu)$*, the resulting Morse family*

$$E^{\bar{H} \to \bar{L}} = \langle \mu, \xi \rangle - h(\mu) \tag{62}$$

generates the Lie-Poisson dynamics.

5 The Reduced Dynamics

5.1 Reduction of Tulczyjew's triplet

Application of the Marsden-Weinstein reduction for the left action of G to the iterated bundles in the trivialized Tulczyjew's triplet results in symplectic projections

$$\begin{aligned}
p_{\ ^1T^*TG} &: \left(^1T^*TG, \Omega_{\ ^1T^*TG} \right) \to \left(\mathfrak{z}_l = \mathcal{O}_\lambda \times \mathfrak{g} \times \mathfrak{g}^*, \Omega_{\mathfrak{z}_l} \right) && (63) \\
&: (g, \xi, \lambda, \nu) \to \left(Ad^*_{g^{-1}} \lambda, \xi, \nu \right), \\
p_{\ ^1T^*T^*G} &: \left(^1T^*T^*G, \Omega_{\ ^1T^*T^*G} \right) \to \left(\mathfrak{z}_d = \mathcal{O}_\lambda \times \mathfrak{g}^* \times \mathfrak{g}, \Omega_{\mathfrak{z}_d} \right) && (64) \\
&: (g, \mu, \lambda, \xi) \to \left(Ad^*_{g^{-1}} \lambda, \mu, \xi \right), \\
p_{\ ^1TT^*G} &: \left(^1TT^*G, \Omega_{\ ^1TT^*G} \right) \to \left(\mathfrak{z}_h = \mathcal{O}_\lambda \times \mathfrak{g}^* \times \mathfrak{g}, \Omega_{\mathfrak{z}_h} \right) && (65) \\
&: (g, \mu, \xi, \nu) \to \left(Ad^*_{g^{-1}} \lambda, \mu, \xi \right),
\end{aligned}$$

into reduced spaces, where $\mathcal{O}_\lambda$ is the coadjoint orbit through $\lambda \in \mathfrak{g}^*$. In the last line, we take the fiber coordinate $v = \lambda - ad^*_\xi \mu$ [14] in order to have convenience in projected coordinates as described by Eqs.(67) and (68) below, as well as in projections in Eqs.(70)-(73).

Remark *In [14], it is shown that, the left action of G on iterated bundles can be trivialized to act on the fiber variables ξ, λ and ν. That means, while performing symplectic quotients, one should consider, literally, the orbits $G_\lambda \backslash (G \times \mathfrak{g} \times \mathfrak{g}^*)$. However, to have a more clear notation, we prefer to take $\mathcal{O}_\lambda \times \mathfrak{g} \times \mathfrak{g}^*$ which is, indeed, diffeomorphic to the correct reduced space [2].*

Following [14], we have the reduced Tulczyjew's triplet

$$
\begin{tikzcd}
\mathcal{O}_\lambda \times \mathfrak{g} \times \mathfrak{g}^* \arrow[dr, "{}^1\pi^{G\backslash}_{G\circledS\mathfrak{g}}"'] & & \mathcal{O}_\lambda \times \mathfrak{g}^* \times \mathfrak{g} \arrow[ll, "{}^1\bar{\sigma}^{G\backslash}_G"'] \arrow[rr, "{}^1\Omega^{G\backslash}_{G\circledS\mathfrak{g}}"] \arrow[dl, "{}^1T\pi^{G\backslash}_G"] \arrow[dr, "{}^1\tau^{G\backslash}_{G\circledS\mathfrak{g}^*}"'] & & \mathcal{O}_\lambda \times \mathfrak{g}^* \times \mathfrak{g} \arrow[dl, "{}^1\pi^{G\backslash}_{G\circledS\mathfrak{g}^*}"] \\
& \mathfrak{g} & & \mathfrak{g}^* &
\end{tikzcd} \tag{66}
$$

consisting of the symplectic diffeomorphisms

$$
\begin{aligned}
\varkappa &: \mathfrak{z}_d \to \mathfrak{z}_l : \left(Ad^*_{g^{-1}}\lambda, \mu, \xi\right) \to \left(Ad^*_{g^{-1}}\lambda, \xi, \mu\right), && (67)\\
\omega^\flat &: \mathfrak{z}_d \to \mathfrak{z}_h : \left(Ad^*_{g^{-1}}\lambda, \mu, \xi\right) \to \left(Ad^*_{g^{-1}}\lambda, \mu, -\xi\right) && (68)
\end{aligned}
$$

obtained from the trivialized symplectic diffeomorphisms ${}^1\bar{\sigma}_G$ and ${}^1\Omega^\flat_{G\circledS\mathfrak{g}^*}$ in Eqs.(40) and (41) by the equations

$$
\varkappa \circ p_{\ {}^1TT^*G} = p_{\ {}^1T^*TG} \circ\ {}^1\bar{\sigma}_G, \quad \text{and} \quad \omega^\flat \circ p_{\ {}^1TT^*G} = p_{\ {}^1T^*TG} \circ\ {}^1\Omega^\flat_{G\circledS\mathfrak{g}^*}. \tag{69}
$$

The projections $\tau_{\mathfrak{z}_l}$, $\tau_{\mathfrak{z}_d}$, $\pi_{\mathfrak{z}_d}$ and $\pi_{\mathfrak{z}_h}$ are trivial

$$
\begin{aligned}
\tau_{\mathfrak{z}_l} &: \mathfrak{z}_l \to \mathfrak{g} : \left(Ad^*_{g^{-1}}\lambda, \xi, \mu\right) \to \xi, && (70)\\
\tau_{\mathfrak{z}_d} &: \mathfrak{z}_d \to \mathfrak{g} : \left(Ad^*_{g^{-1}}\lambda, \mu, \xi, \right) \to \xi, && (71)\\
\pi_{\mathfrak{z}_d} &: \mathfrak{z}_d \to \mathfrak{g}^* : \left(Ad^*_{g^{-1}}\lambda, \mu, \xi\right) \to \mu, && (72)\\
\pi_{\mathfrak{z}_h} &: \mathfrak{z}_h \to \mathfrak{g}^* : \left(Ad^*_{g^{-1}}\lambda, \mu, \xi\right) \to \mu. && (73)
\end{aligned}
$$

In reference [2], Hamiltonian dynamics on $\mathfrak{z}_l$, in connection with those on T^*TG, was studied in detail.

5.2 The Reduced Tulczyjew's Symplectic Space

In order to compute vector fields and one-forms on the reduced Tulczyjew's symplectic space $\mathfrak{z}_d$, we will push the tensor fields on ${}^1TT^*G$ forward by the projection $p_{\,{}^1TT^*G}$. At the point (g,μ,ξ,ν), the tangent mapping of $p_{\,{}^1TT^*G}$ is

$$
\begin{aligned}
T_{(g,\mu,\xi,\nu)}\left(p_{\,{}^1TT^*G}\right) \;&:\; T_{(g,\mu,\xi,\nu)}\left({}^1TT^*G\right) \rightarrow T_{\left(Ad^*_{g^{-1}}\lambda,\mu,\xi\right)}\left(\mathcal{O}_\lambda \times \mathfrak{g}^* \times \mathfrak{g}\right) \\
&:\; (V_g, V_\mu, V_\xi, V_\nu) \rightarrow \left(ad^*_{TR_{g^{-1}}V_g} \circ Ad^*_{g^{-1}}\lambda, V_\mu, V_\xi\right).
\end{aligned}
$$

Pushing a right invariant vector field $X^{{}^1TT^*G}_{(\eta,\upsilon,\zeta,\tilde{\upsilon})}$, in the form given by Eq.(45), forward by $p_{\,{}^1TT^*G}$ we arrive at the vector field

$$
X^{\mathfrak{z}_d}_{(\eta,\upsilon,\zeta)}\left(Ad^*_{g^{-1}}\lambda,\mu,\xi\right) = \left(ad^*_\eta \circ Ad^*_{g^{-1}}\lambda, \upsilon + ad^*_\eta\mu, \zeta + [\xi,\eta]\right) \tag{74}
$$

on $\mathfrak{z}_d$. The Jacobi-Lie bracket of two such vector fields is

$$
\left[X^{\mathfrak{z}_d}_{(\eta,\upsilon,\zeta)}, X^{\mathfrak{z}_d}_{(\bar{\eta},\bar{\upsilon},\bar{\zeta})}\right] = X^{\mathfrak{z}_d}_{\left([\eta,\bar{\eta}],ad^*_{\bar{\eta}}\upsilon - ad^*_\eta\bar{\upsilon},\left[\eta,\bar{\zeta}\right]-[\bar{\eta},\zeta]\right)}. \tag{75}
$$

Proposition *The reduced Tulczyjew's manifold $\mathfrak{z}_d = \mathcal{O}_\lambda \times \mathfrak{g}^* \times \mathfrak{g}$ is an exact symplectic manifold with symplectic two-form $\Omega_{\mathfrak{z}_d}$, potential one-forms χ_1 and χ_2 whose values on vector fields of the form of Eq.(74) at a point $\left(Ad^*_{g^{-1}}\lambda,\mu,\xi\right) \in \mathfrak{z}_d$ are*

$$
\begin{aligned}
\left\langle \Omega_{\mathfrak{z}_d}, \left(X^{\mathfrak{z}_d}_{(\eta,\upsilon,\zeta)}, X^{\mathfrak{z}_d}_{(\bar{\eta},\bar{\upsilon},\bar{\zeta})}\right)\right\rangle \;&=\; \left\langle \upsilon, \bar{\zeta}\right\rangle - \left\langle \bar{\upsilon}, \zeta\right\rangle - \left\langle \lambda, [\eta,\bar{\eta}]\right\rangle, &(76)\\
\left\langle \chi_1, X^{\mathfrak{z}_d}_{(\eta,\upsilon,\zeta)}\right\rangle \left(Ad^*_{g^{-1}}\lambda,\mu,\xi\right) \;&=\; \left\langle \lambda, \eta\right\rangle - \left\langle \upsilon, \xi\right\rangle, &(77)\\
\left\langle \chi_2, X^{\mathfrak{z}_d}_{(\eta,\upsilon,\zeta)}\right\rangle \left(Ad^*_{g^{-1}}\lambda,\mu,\xi\right) \;&=\; \left\langle \lambda, \eta\right\rangle + \left\langle \mu, \zeta\right\rangle, &(78)
\end{aligned}
$$

respectively.

Proof *We recall definitions of the potential one-forms θ_1 and θ_2 in Eqs.(46) and*

(47). Define one-forms χ_1 and χ_2 on $\mathfrak{z}_d$ by the equations

$$
\begin{aligned}
\left\langle \theta_1, X^{\,^1TT^*G}_{(\eta,\upsilon,\zeta,\tilde{\upsilon})} \right\rangle (g,\mu,\xi,\nu) &= \left\langle \chi_1, X^{\mathfrak{z}_d}_{(\eta,\upsilon,\zeta)} \right\rangle \left(Ad^*_{g^{-1}}\lambda, \mu, \xi \right), \\
\left\langle \theta_2, X^{\,^1TT^*G}_{(\eta,\upsilon,\zeta,\tilde{\upsilon})} \right\rangle (g,\mu,\xi,\nu) &= \left\langle \chi_2, X^{\mathfrak{z}_d}_{(\eta,\upsilon,\zeta)} \right\rangle \left(Ad^*_{g^{-1}}\lambda, \mu, \xi \right).
\end{aligned}
$$

The exterior derivative of χ_1 in Eq.(77) gives the symplectic two-form $\Omega_{\mathfrak{z}_d}$. Using the invariant definition of exterior derivative we obtain

$$
\begin{aligned}
\left\langle \Omega_{\mathfrak{z}_d}; \left(X^{\mathfrak{z}_d}_{(\eta,\upsilon,\zeta)}, X^{\mathfrak{z}_d}_{(\bar{\eta},\bar{\upsilon},\bar{\zeta})} \right) \right\rangle &= X^{\mathfrak{z}_d}_{(\eta,\upsilon,\zeta)} \left\langle \chi_1, X^{\mathfrak{z}_d}_{(\bar{\eta},\bar{\upsilon},\bar{\zeta})} \right\rangle - X^{\mathfrak{z}_d}_{(\bar{\eta},\bar{\upsilon},\bar{\zeta})} \left\langle \chi_1, X^{\mathfrak{z}_d}_{(\eta,\upsilon,\zeta)} \right\rangle \\
&\quad - \left\langle \chi_1, \left[X^{\mathfrak{z}_d}_{(\eta,\upsilon,\zeta)}, X^{\mathfrak{z}_d}_{(\bar{\eta},\bar{\upsilon},\bar{\zeta})} \right] \right\rangle \\
&= -\left\langle \bar{\upsilon}, \zeta + [\xi,\eta] \right\rangle - \left(-\left\langle \upsilon, \bar{\zeta} + [\xi,\bar{\eta}] \right\rangle \right) \\
&\quad - \left\langle \lambda, [\eta,\bar{\eta}] \right\rangle - \left\langle ad^*_{\bar{\eta}}\upsilon - ad^*_{\eta}\bar{\upsilon}, \xi \right\rangle \\
&= \left\langle \upsilon, \bar{\zeta} \right\rangle - \left\langle \bar{\upsilon}, \zeta \right\rangle - \left\langle \lambda, [\eta,\bar{\eta}] \right\rangle, \qquad (79)
\end{aligned}
$$

where we used the fact that $\langle \lambda, \eta \rangle$ is a constant for a fixed λ, and the Jacobi Lie bracket in Eq.(75). Similarly, we can show $\Omega_{\mathfrak{z}_d} = d\chi_2$.

It follows from Eqs.(77) and (78) that the difference

$$
\chi_2 - \chi_1 = d\langle \mu, \xi \rangle = d\Delta \qquad (80)
$$

is an exact one-form on $\mathfrak{z}_d$.

The explicit expressions of the reduced symplectic two-forms $\Omega_{\mathfrak{z}_l}$ and $\Omega_{\mathfrak{z}_h}$ on the product bundles $\mathfrak{z}_l$ and $\mathfrak{z}_h$ can be obtained by the pull-back of $\Omega_{\mathfrak{z}_d}$ in Eq.(76) with the symplectic diffeomorphisms $\varkappa$ and $\omega^\flat$ in Eqs.(67) and (68), respectively. The symplectic two-form $\Omega_{\mathfrak{z}_d}$ is an example of the reduced product dynamics defined in proposition 5.4 of [36].

5.3 Euler-Poincaré dynamics as a Lagrangian submanifold

When $\bar{L} = l(\xi)$, the trivialized exterior derivative $^1d\bar{L}$ in Eq.(54) becomes

$$
{}^1dl : \mathfrak{g} \to {}^1T^*TG : \xi \to \left(g, \xi, ad^*_\xi \frac{\delta l}{\delta \xi}, \frac{\delta l}{\delta \xi} \right). \qquad (81)
$$

The image of trivialized exterior derivative ${}^{1}dl$ can be reduced to $\mathfrak{z}_{l}$ by composition with the projection map $p_{\ {}^{1}T^{*}TG}$ in Eq.(63). That is, we define

$$d^{G\backslash}l = p_{\ {}^{1}T^{*}TG} \circ {}^{1}dl : \mathfrak{g} \rightarrow \mathfrak{z}_{l} : \xi \rightarrow \left(ad_{\xi}^{*}\frac{\delta l}{\delta \xi}, \xi, \frac{\delta l}{\delta \xi} \right),$$

where we choose $g = e$ without loss of generality. Applying the inverse $\varkappa^{-1}$ of the symplectic diffeomorphism $\varkappa : \mathfrak{z}_{d} \rightarrow \mathfrak{z}_{l}$ in Eq.(67), we define Lagrange-Dirac derivative

$$\mathfrak{d}l = \varkappa^{-1} \circ d^{G\backslash}l = \varkappa^{-1} \circ p_{\ {}^{1}T^{*}TG} \circ {}^{1}dl : \mathfrak{g} \rightarrow \mathfrak{z}_{d} : \xi \rightarrow \left(ad_{\xi}^{*}\frac{\delta l}{\delta \xi}, \frac{\delta l}{\delta \xi}, \xi \right). \tag{82}$$

Proposition *The image of Lagrange-Dirac derivative $\mathfrak{d}l$, in Eq.(82), is a Lagrangian submanifold $s_{\mathfrak{z}_{d}}$ of $(\mathfrak{z}_{d}, \Omega_{\mathfrak{z}_{d}})$ defining the Euler-Poincaré equations (28).*

Proof *Since, the trivialization map $tr^{1}_{T^{*}T^{*}G}$ is symplectic, the image of ${}^{1}dl$ is a Lagrangian submanifold of ${}^{1}T^{*}TG$. The projection $p_{\ {}^{1}T^{*}TG}$ is symplectic, and hence the image of $d^{G\backslash}l$ is a Lagrangian submanifold $s_{\mathfrak{z}_{l}}$ of $\mathfrak{z}_{l}$. The inverse symplectic diffeomorphism $\varkappa^{-1}$ maps this Lagrangian submanifold $s_{\mathfrak{z}_{l}}$ to a Lagrangian submanifold $s_{\mathfrak{z}_{d}}$ of $\mathfrak{z}_{d}$. So, the image $s_{\mathfrak{z}_{d}}$ of $\mathfrak{d}l$ is a Lagrangian submanifold of $(\mathfrak{z}_{d}, \Omega_{\mathfrak{z}_{d}})$. Under the global trivialization, $s_{\mathfrak{z}_{d}}$ is obtained to be*

$$s_{\mathfrak{z}_{d}} = \left\{ \left(ad_{\xi}^{*}\frac{\delta l}{\delta \xi}, \frac{\delta l}{\delta \xi}, \xi \right) \in \mathfrak{z}_{d} : \xi \in \mathfrak{g} \right\}. \tag{83}$$

When $\bar{L} = l(\xi)$, the Lagrangian submanifold $S_{\ {}^{1}TT^{*}G}$ in Eq.(51) reduces to

$$s_{\ {}^{1}TT^{*}G} = \left\{ \left(e, \frac{\delta l}{\delta \xi}, \xi, 0 \right) \in {}^{1}TT^{*}G : \xi \in \mathfrak{g} \right\},$$

and the first definition in Eq.(69) shows that the projection of $s_{\ {}^{1}TT^{*}G}$ by $p_{\ {}^{1}TT^{*}G}$ is $s_{\mathfrak{z}_{d}}$. The reconstruction mapping ${}^{1}TT^{*}G \rightarrow T(G \circledS \mathfrak{g}^{*})$ in Eq.(52) takes $s_{\ {}^{1}TT^{*}G}$ to the Lagrangian submanifold

$$s_{T(G \circledS \mathfrak{g}^{*})} = \left\{ \left(e, \frac{\delta l}{\delta \xi}; \xi, ad_{\xi}^{*}\frac{\delta l}{\delta \xi} \right) \in T(G \circledS \mathfrak{g}^{*}) : \xi \in \mathfrak{g} \right\}$$

of $T(G \circledS \mathfrak{g}^{*})$, and this defines Euler-Poincaré equations (28).

Alternatively, the formulation that uses the de Rham exterior derivative and the potential one-form χ_2 in Eq.(78) goes as follows.

Proposition *The identity*

$$\tau_{\mathfrak{z}_d}^{*}dl = \chi_2$$

defines the Lagrangian submanifold $s_{\mathfrak{z}_d}$ in Eq.(83), hence the Euler-Poincaré equations (28). Here, $\tau_{\mathfrak{z}_d}$ is the projection $\mathfrak{z}_d \to \mathfrak{g}$, dl is the (de Rham) exterior derivative of l on $\mathfrak{g}$, and χ_2 is the potential one-form in Eq.(78).

Proof *We compute the value of exact one-from $\tau_{\mathfrak{z}_d}^{*}dl = d\left(l \circ \tau_{\mathfrak{z}_d}\right)$ on a vector field $X_{(\eta,\upsilon,\zeta)}^{\mathfrak{z}_d}$ in Eq.(74). At a point $\left(Ad_{g^{-1}}^{*}\lambda, \mu, \xi\right)$, we have*

$$\begin{aligned}
\left\langle \tau_{\mathfrak{z}_d}^{*}dl, X_{(\eta,\upsilon,\zeta)}^{\mathfrak{z}_d}\right\rangle \left(Ad_{g^{-1}}^{*}\lambda, \mu, \xi\right) &= \left\langle dl, \left(\tau_{\mathfrak{z}_d}\right)_{*} X_{(\eta,\upsilon,\zeta)}^{\mathfrak{z}_d}\right\rangle \\
&= \left\langle \frac{\delta l}{\delta \xi}, \zeta + [\xi, \eta]\right\rangle \\
&= \left\langle \frac{\delta l}{\delta \xi}, \zeta\right\rangle + \left\langle ad_{\xi}^{*}\frac{\delta l}{\delta \xi}, \eta\right\rangle,
\end{aligned}$$

where $\left(\tau_{\mathfrak{z}_d}\right)_{*} X_{(\eta,\upsilon,\zeta)}^{\mathfrak{z}_d}$ is the push forward of the vector field $X_{(\eta,\upsilon,\zeta)}^{\mathfrak{z}_d}$ by the projection $\tau_{\mathfrak{z}_d}$ from $\mathfrak{z}_d$ to its third factor $\mathfrak{g}$, that is, to the vector $\zeta + [\xi, \eta]$ in $T_{\xi}\mathfrak{g} \simeq \mathfrak{g}$. Equating this to $\left\langle \chi_2, X_{(\eta,\upsilon,\zeta)}^{\mathfrak{z}_d}\right\rangle$ in Eq.(78) gives the Lagrangian submanifold $s_{\mathfrak{z}_d} = im\left(\mathfrak{d}l\right)$ in Eq.(83) via

$$\lambda = ad_{\xi}^{*}\frac{\delta l}{\delta \xi} \quad \text{and} \quad \mu = \frac{\delta l}{\delta \xi}$$

in coordinates (λ, μ, ξ) of $\mathfrak{z}_d$.

5.4 Lie-Poisson dynamics as a Lagrangian submanifold

Consider a Hamiltonian function $\bar{H}$ on $G \circledS \mathfrak{g}^{*}$ and define $h : \mathfrak{g}^{*} \to \mathbb{R}$ by $\bar{H} = h(\mu)$. With the trivialized exterior derivative

$$-\ ^{1}dh : \mathfrak{g}^{*} \to\ ^{1}T^{*}T^{*}G : (g, \mu) \to \left(g, \mu, ad_{\frac{\delta h}{\delta \mu}}^{*}\mu, -\frac{\delta h}{\delta \mu}\right),$$

and the projection $p_{\ ^{1}T^{*}T^{*}G} :\ ^{1}T^{*}T^{*}G \to \mathfrak{z}_h$ in Eq.(64), we define

$$-d^{G\backslash}h = p_{\ ^{1}T^{*}T^{*}G} \circ\ ^{1}d\left(-h\right) : \mathfrak{g}^{*} \to \mathfrak{z}_h : \mu \to \left(ad_{\frac{\delta h}{\delta \mu}}^{*}\mu, \mu, -\frac{\delta h}{\delta \mu}\right)$$

by choosing $g = e$. Applying the inverse $\omega^\sharp$ of the symplectic diffeomorphism $\omega^\flat$ in Eq.(68), we obtain the Hamilton-Dirac derivative

$$-\,\mathfrak{d}h = \omega^\sharp \circ d^{G\backslash}(-h) : \mathfrak{g}^* \to \mathfrak{z}_d : \mu \to \left(ad^*_{\frac{\delta h}{\delta \mu}}\mu, \mu, \frac{\delta h}{\delta \mu} \right). \tag{84}$$

Proposition *The image of the Hamilton-Dirac derivative $-\mathfrak{d}h$ is a Lagrangian submanifold $s'_{\mathfrak{z}_d}$ of $(\mathfrak{z}_d, \Omega_{\mathfrak{z}_d})$ and it defines the Lie-Poisson equations (35).*

Proof *The image of $-\,{}^1dh$ is a Lagrangian submanifold of ${}^1T^*T^*G$ and $p_{\,{}^1T^*T^*G}$ maps this Lagrangian submanifold to a Lagrangian submanifold $s'_{\mathfrak{z}_h}$ of $\mathfrak{z}_h$. The musical isomorphism $\omega^\sharp$ takes $s'_{\mathfrak{z}_h}$ to the Lagrangian submanifold $s'_{\mathfrak{z}_d}$ of $(\mathfrak{z}_d, \Omega_{\mathfrak{z}_d})$. Thus,*

$$s'_{\mathfrak{z}_d} = im\,(-\mathfrak{d}h) = \omega^\sharp \circ im\left(d^{G\backslash}(-h)\right) = \omega^\sharp \circ p_{\,{}^1T^*T^*G} \circ im\left(-\,{}^1dh\right). \tag{85}$$

From Eqs. (64) and (65) we have

$$\omega^\sharp \circ p_{\,{}^1T^*T^*G} = p_{\,{}^1TT^*G} \circ\, {}^1\Omega^\sharp_{G \circledS \mathfrak{g}^*},$$

where ${}^1\Omega^\sharp_{G \circledS \mathfrak{g}^*}$ is the inverse of the isomorphism ${}^1\Omega^\flat_{G \circledS \mathfrak{g}^*}$ in Eq.(41). This implies that $s'_{\mathfrak{z}_d}$ is the projection $p_{\,{}^1TT^*G}\left(s'_{\,{}^1TT^*G}\right)$ of the Lagrangian submanifold

$$s'_{\,{}^1TT^*G} = \left\{ \left(g, \mu; \frac{\delta h}{\delta \mu}, 0 \right) \in {}^1TT^*G : \mu \in \mathfrak{g}^* \right\}$$

obtained from $S'_{\,{}^1TT^*G}$ in Eq.(56) by substituting $\bar{H} = h(\mu)$. The reconstruction mapping ${}^1TT^*G \to T(G \circledS \mathfrak{g}^*)$ in the Eq.(52) takes $s'_{\,{}^1TT^*G}$ to the Lagrangian submanifold

$$s'_{T(G \circledS \mathfrak{g}^*)} = \left\{ \left(g, \mu; TR_g \frac{\delta h}{\delta \mu}, ad^*_{\frac{\delta h}{\delta \mu}}\mu \right) \in T(G \circledS \mathfrak{g}^*) : \mu \in \mathfrak{g}^* \right\}$$

of $T(G \circledS \mathfrak{g}^*)$ which is the Lie-Poisson equation (33) equivalent to (35).

Alternatively, with exterior derivative and the potential one-form χ_1 in Eq.(77), we have

Proposition *The equation*

$$-\pi^*_{\mathfrak{z}_d} dh = \chi_1$$

defines the Lagrangian submanifold $s'_{\mathfrak{z}_d}$ in Eq.(85) and gives the Lie-Poisson equations (35). Here, $\pi_{\mathfrak{z}_d}$ is the projection $\mathfrak{z}_d \to \mathfrak{g}^*$, dh is the (de Rham) exterior derivative of h on $\mathfrak{g}^*$, and χ_1 is the potential one-form in Eq.(77).

Proof *To prove this identity, we compute the value of exact one-from $\pi^*_{\mathfrak{z}_d} dh = d(h \circ \pi_{\mathfrak{z}_d})$ on a vector field $X^{\mathfrak{z}_d}_{(\eta,\upsilon,\zeta)}$ over $\mathfrak{z}_d$. At a point $\left(Ad^*_{g^{-1}}\lambda, \mu, \xi\right)$, we have*

$$
\begin{aligned}
\left\langle -\pi^*_{\mathfrak{z}_d} dh, X^{\mathfrak{z}_d}_{(\eta,\upsilon,\zeta)} \right\rangle \left(Ad^*_{g^{-1}}\lambda, \mu, \xi\right) &= -\left\langle dh, (\pi_{\mathfrak{z}_d})_* X^{\mathfrak{z}_d}_{(\eta,\upsilon,\zeta)} \right\rangle \\
&= -\left\langle \frac{\delta h}{\delta \mu}, \upsilon + ad^*_{\eta}\mu \right\rangle \\
&= -\left\langle \frac{\delta h}{\delta \mu}, \upsilon \right\rangle + \left\langle ad^*_{\frac{\delta h}{\delta \mu}}\mu, \eta \right\rangle,
\end{aligned}
$$

where $(\pi_{\mathfrak{z}_d})_* X^{\mathfrak{z}_d}_{(\eta,\upsilon,\zeta)}$ is the push forward of the vector field $X^{\mathfrak{z}_d}_{(\eta,\upsilon,\zeta)}$ by the projection $\pi_{\mathfrak{z}_d}$ from $\mathfrak{z}_d$ to its second factor $\mathfrak{g}^*$, that is to the dual vector $\upsilon + ad^*_{\eta}\mu$ in $T_\mu \mathfrak{g}^* \simeq \mathfrak{g}^*$. Equating this to $\left\langle \chi_1, X^{\mathfrak{z}_d}_{(\eta,\upsilon,\zeta)} \right\rangle$ in Eq.(77) defines the Lagrangian submanifold $s'_{\mathfrak{z}_d} = im(\mathfrak{d}h)$ in Eq.(85) given in coordinates (λ, μ, ξ) of $\mathfrak{z}_d$ by

$$
\lambda = ad^*_{\frac{\delta h}{\delta \mu}}\mu \quad \text{and} \quad \xi = \frac{\delta h}{\delta \mu}.
$$

5.5 Legendre Transformation for Reduced Dynamics

Being cotangent bundles, $T^*\mathfrak{g} = \mathfrak{g} \times \mathfrak{g}^*$ and $T^*\mathfrak{g}^* = \mathfrak{g}^* \times \mathfrak{g}$ are canonically symplectic. It is possible to embed $T^*\mathfrak{g}$ and $T^*\mathfrak{g}^*$ symplectically into the total space $\mathfrak{z}_d$

$$
\begin{aligned}
\hat{\varkappa} &: T^*\mathfrak{g} \to \mathfrak{z}_d : (\xi, \mu) \to \left(ad^*_{\xi}\mu, \mu, \xi\right), && (86) \\
\hat{\omega} &: T^*\mathfrak{g}^* \to \mathfrak{z}_d : (\mu, \xi) \to \left(ad^*_{\xi}\mu, \mu, \xi\right). && (87)
\end{aligned}
$$

The following proposition shows how to define Lagrangian submanifold $im(\mathfrak{d}l) = s_{\mathfrak{z}_d}$ in Eq.(83) from the right wing (that is from the Hamiltonian side) of the reduced Tulczyjew's triplet (66).

Proposition *The Lagrangian dynamics determined by the Lagrangian submani-*

fold $s_{\mathfrak{z}_d}$ in Eq.(83) is generated by the Morse family

$$E^{l\to h} = (l\circ\tau_{\mathfrak{z}_d}) + \Delta = l(\xi) - \langle\mu,\xi\rangle \tag{88}$$

on the bundle $\mathfrak{g}\times\mathfrak{g}^*\to\mathfrak{g}^*$. Here, Δ is a real valued function on $\mathfrak{g}\times\mathfrak{g}^*$ obtained from the equation

$$\chi_2 - \chi_1 = d\Delta = d\langle\mu,\xi\rangle$$

where χ_1 and χ_2 are given in Eqs.(77) and (78), respectively.

Proof *In the trivialized cases, the Legendre transformations have been achieved by Morse families on the trivialized Pontryagin bundle $G\circledS(\mathfrak{g}\times\mathfrak{g}^*)$. For the reduced dynamics, due to the invariance under the group action G, the Morse families will be defined on $G\backslash(G\circledS(\mathfrak{g}\times\mathfrak{g}^*))\simeq\mathfrak{g}\times\mathfrak{g}^*$. Recall the generating object*

$$\begin{tikzcd} \mathfrak{z}_d \arrow[drr, "\pi_{\mathfrak{z}_d}"] & & \mathfrak{g}^*\times\mathfrak{g} \arrow[ll, "\hat{\omega}"'] \arrow[d] \\ & & \mathfrak{g}^* \end{tikzcd}$$

where $\hat{\omega}$ is the embedding in Eq.(87). According to general theory of generating families (c.f. Eq.(6)), the Morse family $E^{l\to h}$ generates a Lagrangian submanifold $s_{T^*\mathfrak{g}^*}$ of $T^*\mathfrak{g}^*$ given by

$$s_{T^*\mathfrak{g}^*} = \left\{(\mu,\xi)\in T^*\mathfrak{g}^* : T^*\pi_{\mathfrak{z}_d}(\mu,\xi) = dE^{l\to h}(\mu,\xi)\right\}. \tag{89}$$

Explicitly, the Lagrangian submanifold $s_{T^*\mathfrak{g}^*}$ consists of two-tuples $(\delta l/\delta\xi,\xi)$. Hence, the image of $s_{T^*\mathfrak{g}^*}$ under the map $\hat{\omega}$ is $s_{\mathfrak{z}_d}$. When the Lagrangian l is not regular then it is not possible to define a function h on $\mathfrak{g}^*$ generating $s_{\mathfrak{z}_d}$. In this case, we only have Morse family $E^{l\to h}$.

Remark *Recall that, the Morse family $E^{\bar{L}\to\bar{H}}$, defined in Eq.(60), is also a generating family for Euler-Poincaré equations. Here, the function $E^{\bar{L}\to\bar{H}}$ is defined on the Pontryagin bundle ${}^1PG$ with base manifold $G\circledS\mathfrak{g}^*$ whereas $E^{l\to h}$, in Eq.(88), is defined on $T^*\mathfrak{g}^*$ with cotangent bundle projection.*

Now, we will establish the inverse Legendre transformation. The Hamiltonian dynamics is defined by a Hamiltonian function h on $\mathfrak{g}^*$. A Hamiltonian functional on $\mathfrak{g}^*$ defines the Lagrangian submanifold $s'_{\mathfrak{z}_d}$, in Eq.(85), of $(\mathfrak{z}_d,\Omega_{\mathfrak{z}_d})$. The

following proposition shows how to generate $s'_{\mathfrak{z}_d}$ using the left wing (that is the Lagrangian side) of the reduced Tulczyjew's triplet (66).

Proposition *The Morse family*

$$E^{h\to l} = \Delta - h(\mu) = \langle \mu, \xi \rangle - h(\mu) \tag{90}$$

on the bundle $\mathfrak{g} \times \mathfrak{g}^* \to \mathfrak{g}$ generates the Lagrangian submanifold $s'_{\mathfrak{z}_d}$ in Eq.(85).

Proof *In this case, diagram is*

$$\begin{tikzcd}
\mathfrak{g}^* \times \mathfrak{g} \arrow[r, "\hat{\varkappa}"] \arrow[d] & \mathfrak{z}_d \arrow[dl, "\tau_{\mathfrak{z}_d}"] \\
\mathfrak{g} &
\end{tikzcd}$$

where $\hat{\varkappa}$ is the embedding, in Eq.(86), of $T^*\mathfrak{g}$ into $\mathfrak{z}_d$. The Morse family $E^{h\to l}$ in Eq.(90) generates a Lagrangian submanifold

$$s_{T^*\mathfrak{g}} = \left\{ (\xi, \mu) \in T^*\mathfrak{g} : T^*\tau_{\mathfrak{z}_d}(\xi, \mu) = dE^{h\to l}(\xi, \mu) \right\}, \tag{91}$$

of $T^*\mathfrak{g}$ (c.f. Eq.(6)). This Lagrangian submanifold consists of two-tuples $(\delta h/\delta\mu, \mu)$. $\hat{\varkappa}$ maps $s_{T^*\mathfrak{g}}$ to $s'_{\mathfrak{z}_d}$. When the Hamiltonian h is not regular then it is not possible to find a Lagrangian function l on $\mathfrak{g}$. In this case, we only have Morse family $E^{h\to l}$.

Remark *Recall that, the Morse family $E^{\bar{H}\to\bar{L}}$, defined in Eq.(62), is another generating family for Lie-Poisson dynamics. $E^{\bar{H}\to\bar{L}}$ is defined on the Pontryagin bundle ${}^1PG$ with base manifold $G \circledS \mathfrak{g}$ whereas $E^{l\to h}$, in Eq.(90), is defined on $T^*\mathfrak{g}$ with tangent bundle projection.*

To summarize, in order to use the classical formulations of generating objects, we employ the following reduced form of Tulczyjew's triplet

$$\begin{tikzcd}
T^*\mathfrak{g} \arrow[r, "\hat{\varkappa}"] \arrow[dr] & \mathfrak{z}_d \arrow[d, "\tau_{\mathfrak{z}_d}"'] \arrow[dr, "\pi_{\mathfrak{z}_d}"] & T^*\mathfrak{g}* \arrow[l, "\hat{\omega}"'] \arrow[d] \\
& \mathfrak{g} & \mathfrak{g}^*
\end{tikzcd}$$

where we replace the total spaces $\mathfrak{z}_l$ and $\mathfrak{z}_h$ by $T^*\mathfrak{g}$ and $T^*\mathfrak{g}^*$, respectively. The payoff is that the mappings $\hat{\varkappa}$ and $\hat{\omega}$ are symplectic embeddings but not isomorphisms.

6 Example: Diffeomorphism Groups

6.1 Group Structure

Group $\mathcal{D}$ of diffeomorphisms on $\mathcal{Q}$ is a Lie group (see for example [4, 13, 29]). Lie algebra of $\mathcal{D}$ is the space $\mathfrak{X}$ of vector fields on $\mathcal{Q}$. The (right) adjoint action Ad of $\mathcal{D}$ on $\mathfrak{X}$ is given by the pull-back operation φ^*X, for $\varphi \in G$ and $X \in \mathfrak{X}$. The infinitesimal adjoint action of an element $Y \in \mathfrak{X}$ on $X \in \mathfrak{X}$ is the Lie derivative of X in the direction of Y, that is $\mathcal{L}_Y X$. The tangent space

$$T_\varphi \mathcal{D} = \{X_\varphi : \mathcal{Q} \to T\mathcal{Q} : X_\varphi = X \circ \varphi \text{ for some } X \in \mathfrak{X}\}$$

at $\varphi \in \mathcal{D}$ consists of material velocity fields. The lifted group multiplication on the tangent bundle $T\mathcal{D}$ is

$$\varpi_{T\mathcal{D}} (X_\varphi, Y_\psi) = X_{\varphi\circ\psi} + T\varphi \circ Y_\psi. \tag{92}$$

The right and the left trivializations of $T\mathcal{D}$ are

$$\begin{aligned} tr^R_{T\mathcal{D}} &: T\mathcal{D} \to \mathcal{D} \times \mathfrak{X} : X_\varphi \to (\varphi, X) \\ tr^L_{T\mathcal{D}} &: T\mathcal{D} \to \mathcal{D} \times \mathfrak{X} : X_\varphi \to (\varphi, \varphi^* X). \end{aligned} \tag{93}$$

After choosing the right trivialization $tr^R_{T\mathcal{D}}$, we arrive at the semidirect product group multiplication

$$\varpi_{\mathcal{D}\circledS\mathfrak{X}} ((\varphi, X), (\psi, Y)) = (\varphi\psi, X + \varphi_* Y) \tag{94}$$

on $\mathcal{D}\circledS\mathfrak{X}$. The Lie algebra of $\mathcal{D}\circledS\mathfrak{X}$ is $\mathfrak{X}\circledS\mathfrak{X}$ with semi-direct product

$$[(X_1, X_2), (Y_1, Y_2)]_{\mathfrak{X}\circledS\mathfrak{X}} = ([X_1, Y_1], [X_1, Y_2] - [Y_1, X_2,]).$$

The dual space $\mathfrak{X}^*$ of the Lie algebra $\mathfrak{X}$ is the space $\Lambda^1(\mathcal{Q}) \otimes Den(\mathcal{Q})$ of one-

form densities on $\mathcal{Q}$. The pairing between $\mu \otimes d^n q \in \mathfrak{X}^*$ and $X \in \mathfrak{X}$ is given by the integration

$$\langle \mu \otimes d^n q, X \rangle = \int_{\mathcal{M}} \langle \mu, X \rangle_{\mathcal{Q}} \, d^n q, \tag{95}$$

where $d^n q$ is the top-form on $\mathcal{Q}$. The pairing inside the integral is the natural pairing of finite dimensional spaces $T_q\mathcal{Q}$ and $T_q^*\mathcal{Q}$. The coadjoint action Ad^* of $\mathcal{D}$ on $\mathfrak{X}^*$ is the pull-back operation $\varphi^* (\mu \otimes d^n q)$ for $\varphi \in \mathcal{D}$ and $\mu \otimes d^n q \in \mathfrak{X}^*$. The infinitesimal coadjoint action ad^* of an element $X \in \mathfrak{X}$ on $\mu \otimes d^n q \in \mathfrak{X}^*$ is minus the Lie derivative of $\mu \otimes d^n q$ by X, that is

$$ad_X^* : \mathfrak{X}^* \to \mathfrak{X}^* : \mu \otimes d^n q \to - \left(\mathcal{L}_X \mu + \left(div_{d^n q} X \right) \mu \right) \otimes d^n q. \tag{96}$$

Here, $div_{d^n q} X$ denotes divergence of the vector field X with respect to the top-form $d^n q$. The cotangent space at φ is

$$T_\varphi^* \mathcal{D} = \left\{ \left(\mu_\varphi : \mathcal{Q} \to T^*\mathcal{Q} \right) \otimes d^n q : \mu_\varphi = \mu \circ \varphi, \, \mu \in \Lambda^1 \left(\mathcal{Q} \right) \right\}.$$

The pairing between $T_\varphi^*\mathcal{D}$ and $T_\varphi\mathcal{D}$ is taken to be the right invariant L^2-integral. Cotangent lifts of right and left actions of $\mathcal{D}$ on $T^*\mathcal{D}$ can be computed using

$$T_{\varphi\circ\psi}^* R_{\psi^{-1}} \left(\mu_\varphi \right) = \mu_{\varphi\circ\psi}, \quad T_{\psi\circ\varphi}^* L_{\psi^{-1}} \mu_\varphi = T^*\psi^{-1} \circ \mu_\varphi,$$

respectively. The cotangent bundle $T^*\mathcal{D}$ is a Lie group with the group multiplication

$$\left(\mu_\varphi, \nu_\psi \right) = \mu_{\varphi\circ\psi} + T^*\varphi^{-1} \circ \nu_\psi.$$

The right and left trivializations of $T^*\mathcal{D}$ are

$$\begin{aligned} tr_{T^*\mathcal{D}}^R &: T^*\mathcal{D} \to \mathcal{D} \times \mathfrak{X}^* : \mu_\varphi \otimes d^n q \to \left(\varphi, \mu \otimes d^n q \right) \\ tr_{T^*\mathcal{D}}^L &: T^*\mathcal{D} \to \mathcal{D} \times \mathfrak{X}^* : \mu_\varphi \otimes d^n q \to \left(\varphi, \varphi^*\mu \otimes \varphi^* d^n q \right). \end{aligned} \tag{97}$$

We choose the right trivialization to arrive at the semi-direct product group structure with multiplication

$$\varpi_{\mathcal{D} \circledS \mathfrak{X}^*} \left(\left(\varphi, \mu \otimes d^n q \right), \left(\psi, \nu \otimes d^n q \right) \right) = \left(\varphi \circ \psi, \left(\mu + \varphi_* \nu \right) \otimes \varphi_* d^n q \right)$$

on the trivialization $\mathcal{D}Ⓢ\mathfrak{X}^*$.

6.2 The Trivialized Dynamics

Let $\bar{L} = \bar{L}(\varphi, X)$ be a Lagrangian density on $\mathcal{D}Ⓢ\mathfrak{X}$, the trivialized Euler-Lagrange equations are

$$\frac{d}{dt}\frac{\delta \bar{L}}{\delta X} = \frac{\delta \bar{L}}{\delta \varphi} \circ \varphi^{-1} - \mathcal{L}_X \frac{\delta \bar{L}}{\delta X} - (div_{d^n q} X) \frac{\delta \bar{L}}{\delta X}. \tag{98}$$

$\bar{L}$ generates a Lagrangian submanifold

$$S_{{}^1TT^*\mathcal{D}} = \left(\varphi, \frac{\delta \bar{L}}{\delta X}, X, \frac{\delta \bar{L}}{\delta \varphi} \circ \varphi^{-1}\right) \tag{99}$$

of the trivialized Tulczyjew's symplectic space ${}^1TT^*\mathcal{D}$ defined by the semi-direct product $(\mathcal{D}Ⓢ\mathfrak{X}^*)Ⓢ(\mathfrak{X}Ⓢ\mathfrak{X}^*)$. Here, the trivialization map

$$T(\mathcal{D}Ⓢ\mathfrak{X}^*) \to {}^1TT^*\mathcal{D} : (X_\varphi, Y_\mu) \to (\varphi, \mu \otimes d^n q, X, Y_\mu + \mathcal{L}_X \mu + (div_{d^n q} X)\, \mu \otimes d^n q) \tag{100}$$

realizes the relation between the Lagrangian submanifold $S_{{}^1TT^*\mathcal{D}}$ and the trivialized Euler-Lagrange equation (98). The Legendre transformation of trivialized Euler-Lagrange equation can be achieved by the Morse family

$$E(\varphi, \mu \otimes d^n q, X) = \bar{L}(\varphi, X) - \int_{\mathcal{Q}} \langle \mu, X \rangle_{\mathcal{Q}}\, d^n q$$

on the Pontryagin bundle $\mathcal{D}Ⓢ(\mathfrak{X}^* \oplus \mathfrak{X})$ over $\mathcal{D}Ⓢ\mathfrak{X}^*$.

A right invariant vector field on $\mathcal{D}Ⓢ\mathfrak{X}^*$ is given by

$$X_{(X,\nu)}^{\mathcal{D}Ⓢ\mathfrak{X}^*}(\varphi, \mu) = (X_\varphi, \nu - \mathcal{L}_X \mu - (div_{d^n q} X)\, \mu \otimes d^n q).$$

At $(\varphi, \mu \otimes d^n q)$, the values of canonical one-form $\theta_{\mathcal{D}Ⓢ\mathfrak{X}^*}$ and the symplectic two-form $\Omega_{\mathcal{D}Ⓢ\mathfrak{X}^*}$ on the right invariant vector fields are

$$\begin{aligned} \left\langle \theta_{\mathcal{D}Ⓢ\mathfrak{X}^*}, X_{(X,\nu)}^{\mathcal{D}Ⓢ\mathfrak{X}^*} \right\rangle &= \int_{\mathcal{Q}} \langle \mu, X \rangle_{\mathcal{Q}}\, d^n q, \\ \left\langle \Omega_{\mathcal{D}Ⓢ\mathfrak{X}^*}; \left(X_{(X,\nu)}^{\mathcal{D}Ⓢ\mathfrak{X}^*}, X_{(Y,\lambda)}^{\mathcal{D}Ⓢ\mathfrak{X}^*} \right) \right\rangle &= \int_{\mathcal{Q}} \langle \nu, Y \rangle_{\mathcal{Q}} - \langle \lambda, X \rangle_{\mathcal{Q}} + \langle \mu, [X, Y] \rangle_{\mathcal{Q}}\, d^n q. \end{aligned}$$

For a Hamiltonian function $\bar{H}$ on $\mathcal{D}Ⓢ\mathfrak{X}^*$, the trivialized Hamilton's equations are

$$\frac{d\varphi}{dt} = \left(\frac{\delta \bar{H}}{\delta \mu}\right)_{\varphi}, \quad \frac{d\mu}{dt} = -\mathcal{L}_{\frac{\delta \bar{H}}{\delta \mu}}\mu - \left(div_{d^n q}\frac{\delta \bar{H}}{\delta \mu}\right)\mu - \left(\frac{\delta \bar{H}}{\delta \varphi}\right)\circ \varphi^{-1}, \tag{101}$$

where, due to the reflexivity assumption, $\delta \bar{H}/\delta\mu$ is assumed to be a vector field in the Lie algebra, and $\left(\delta \bar{H}/\delta \mu\right)_{\varphi}$ is the material velocity field. The Lagrangian submanifold generated by the Hamiltonian function $\bar{H}$ is

$$S'_{{}^1TT^*\mathcal{D}} = \left(\varphi, \mu \otimes d^n q, \frac{\delta \bar{H}}{\delta \mu}, -\frac{\delta \bar{H}}{\delta \varphi}\circ \varphi^{-1}\otimes d^n q\right).$$

To establish the link between $S'_{{}^1TT^*\mathcal{D}}$ and Eq.(101), we refer to the trivialization map (100). The Legendre transformation of the Hamiltonian dynamics described by Eqs.(101) results from the Morse family

$$E\left(\varphi, \mu \otimes d^n q, X\right) = \int_{\mathcal{Q}} \langle \mu, X\rangle_{\mathcal{Q}}\, d^n q - \bar{H}\left(\varphi, \mu\right)$$

on the Pontryagin bundle $\mathcal{D}Ⓢ\left(\mathfrak{X}^* \oplus \mathfrak{X}\right)$ over $\mathcal{D}Ⓢ\mathfrak{X}$.

6.3 The Reduced Dynamics

When the Lagrangian $\bar{L}$ is free of the group variable, we have $\bar{L} = l\left(X\right)$ and the trivialized Euler-Lagrange equations (98) reduces to the Euler-Poincaré equations

$$\frac{d}{dt}\frac{\delta l}{\delta X} = -\mathcal{L}_X \frac{\delta l}{\delta X} - \left(div_{d^n q} X\right)\frac{\delta l}{\delta X}. \tag{102}$$

Similarly, when the Hamiltonian depends on the fiber variable μ only, $\bar{H}\left(g, \mu\right) = h\left(\mu\right)$, the trivialized Hamilton's equation (101) gives the Lie-Poisson equation

$$\frac{d\mu}{dt} = -\mathcal{L}_{\frac{\delta h}{\delta \mu}}\mu - \left(div_{d^n q}\frac{\delta h}{\delta \mu}\right)\mu. \tag{103}$$

In order to perform the Legendre transformations of the reduced dynamics, we present them as Lagrangian submanifolds of the reduced Tulczyjew's symplectic space $\mathcal{O}_\lambda \times \mathfrak{X}^* \times \mathfrak{X}$. Here, $\mathcal{O}_\lambda \times \mathfrak{X}^* \times \mathfrak{X}$ is obtained by application of the Marsden-Weinstein reduction to the trivialized Tulczyjew's symplectic space ${}^1TT^*\mathcal{D}$, that

is

$$
{}^{1}TT^{*}\mathcal{D} \to \mathcal{O}_{\lambda} \times \mathfrak{X}^{*} \times \mathfrak{X} : (\varphi, \mu \otimes d^{n}q, X, \nu \otimes d^{n}q) \to (\varphi_{*} (\lambda \otimes d^{n}q), \mu \otimes d^{n}q, X), \tag{104}
$$

where $\lambda = \nu - \mathcal{L}_X \mu - (div_{d^n q} X)\, \mu$. For a Lagrangian l on $\mathfrak{X}$, image of the Lagrange-Dirac derivative

$$
\mathfrak{d}l : \mathfrak{X} \to \mathcal{O}_{\lambda} \times \mathfrak{X}^{*} \times \mathfrak{X} : X \to \left(-\mathcal{L}_X \frac{\delta l}{\delta X} - (div_{d^n q} X) \frac{\delta l}{\delta X} \otimes d^{n}q, \frac{\delta l}{\delta X} \otimes d^{n}q, X \right)
$$

is a Lagrangian submanifold of $\mathcal{O}_{\lambda} \times \mathfrak{X}^{*} \times \mathfrak{X}$. The image $im\,(\mathfrak{d}l)$ determines Euler-Poincaré equations. The Legendre transformation is generated by the Morse family

$$
E^{l \to h} (\mu \otimes d^{n}q, X) = l\,(X) - \int_{\mathcal{Q}} \langle \mu, X \rangle_{\mathcal{Q}}\, d^{n}q
$$

on the bundle $\mathfrak{X}^{*} \times \mathfrak{X} \to \mathfrak{X}^{*}$. Similarly, for a Hamiltonian h on $\mathfrak{X}^{*}$, the image of Hamilton-Dirac derivative

$$
-\mathfrak{d}h : \mathfrak{X}^{*} \to \mathcal{O}_{\lambda} \times \mathfrak{X}^{*} \times \mathfrak{X} : \mu \to \left(-\mathcal{L}_{\frac{\delta h}{\delta \mu}} \mu - \left(div_{d^n q} \frac{\delta h}{\delta \mu} \right) \mu \otimes d^{n}q, \mu \otimes d^{n}q, \frac{\delta h}{\delta \mu} \right)
$$

is a Lagrangian submanifold of $\mathcal{O}_{\lambda} \times \mathfrak{X}^{*} \times \mathfrak{X}$ and determines Lie-Poisson equations. The inverse Legendre transformation of the Euler-Poincaré dynamics is generated by the Morse family

$$
E^{h \to l} (\mu \otimes d^{n}q, X) = \int_{\mathcal{Q}} \langle \mu, X \rangle_{\mathcal{Q}}\, d^{n}q - h\,(\mu) \tag{105}
$$

on the bundle $\mathfrak{X}^{*} \times \mathfrak{X} \to \mathfrak{X}$.

7 Summary, Discussions and Prospectives

We obtain trivialized and reduced dynamics for Hamiltonian and Lagrangian formulations of systems with configuration space G. Following diagram summarizes

these equations and their representations by Lagrangian submanifolds.

	Dynamics	Lagrangian Submanifold
Trivialized Euler-Lagrange Equations on $G \circledS \mathfrak{g}$	$\frac{d}{dt}\frac{\delta \bar{L}}{\delta \xi} = T_e^* R_g \frac{\delta \bar{L}}{\delta g} + ad_\xi^* \frac{\delta \bar{L}}{\delta \xi}$	$\left\{ g, \frac{\delta \bar{L}}{\delta \xi}, \xi, T^* R_g \frac{\delta \bar{L}}{\delta g} \right\} \subset {}^1TT^*G$
Trivialized Hamilton's Equations on $G \circledS \mathfrak{g}^*$	$\frac{dg}{dt} = T_e R_g \left(\frac{\delta \bar{H}}{\delta \mu} \right)$, $\frac{d\mu}{dt} = ad^*_{\frac{\delta \bar{H}}{\delta \mu}} \mu - T_e^* R_g \frac{\delta \bar{H}}{\delta g}$	$\left\{ g, \mu, \frac{\delta \bar{H}}{\delta \mu}, -T^* R_g \frac{\delta \bar{H}}{\delta g} \right\} \subset {}^1TT^*G$
Euler-Poincaré Equations on $\mathfrak{g}$	$ad_\xi^* \frac{\delta l}{\delta \xi} - \frac{d}{dt}\frac{\delta l}{\delta \xi} = 0$	$\left\{ ad_\xi^* \frac{\delta l}{\delta \xi}, \frac{\delta l}{\delta \xi}, \xi \right\} \subset \mathfrak{z}_d$
Lie-Poisson Equations on $\mathfrak{g}^*$	$\frac{d\mu}{dt} = ad^*_{\frac{\delta h}{\delta \mu}} \mu$	$\left\{ ad^*_{\frac{\delta h}{\delta \mu}} \mu, \mu, \frac{\delta h}{\delta \mu} \right\} \subset \mathfrak{z}_d$

We identify the following Morse families for trivialized and reduced dynamics.

	Morse family	Bundle
Trivialized Euler-Lagrange Equations on $G \circledS \mathfrak{g}$	$E^{\bar{L} \to \bar{H}} (g, \xi, \mu) = \bar{L}(g, \xi) - \langle \mu, \xi \rangle$	$G \circledS (\mathfrak{g} \times \mathfrak{g}^*) \to G \circledS \mathfrak{g}^*$
Trivialized Hamilton's Equations on $G \circledS \mathfrak{g}^*$	$E^{\bar{H} \to \bar{L}} (g, \xi, \mu) = \langle \mu, \xi \rangle - \bar{H}(g, \mu)$	$G \circledS (\mathfrak{g} \times \mathfrak{g}^*) \to G \circledS \mathfrak{g}$
Euler-Poincaré Equations on $\mathfrak{g}$	$E^{l \to h} (\xi, \mu) = l(\xi) - \langle \mu, \xi \rangle$	$\mathfrak{g} \times \mathfrak{g}^* \to \mathfrak{g}^*$
Lie-Poisson Equations on $\mathfrak{g}^*$	$E^{h \to l} (\xi, \mu) = \langle \mu, \xi \rangle - h(\mu)$	$\mathfrak{g} \times \mathfrak{g}^* \to \mathfrak{g}$.

Obviously, the form of dynamical equations obtained in this work depends on the trivialization we employed. What we refer to trivialization of the first kind carries the group operations to iterated bundles and contributes additional term due to semi-direct product structures. Higher order dynamics on Lie groups with adapted trivializations of higher order and iterated bundles as well as their symplectic and Poisson reductions are under investigation [12].

The reduction of Tulczyjew's symplectic space can be generalized to symplectic reduction of tangent bundle of a symplectic manifold with the lifted symplectic structure. This could be the first step toward more general studies on the reduction of the special symplectic structures and the reduction of Tulczyjew's triplet with configuration manifold $\mathcal{Q}$.

Finally, we want to mention that the foremost example of degenerate system that falls into application area of present formulation is the Vlasov-Poisson equation of plasma dynamics which was, indeed, the motivation for this work.

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